Inverse-free subreducts of lattice-ordered groups

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January, 2020

A *lattice-ordered group*, or *ℓ-group*, is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, ^{-1}, 1)$ such that

- $(A, \wedge, \vee)$ is a lattice,
- $(A, \cdot, ^{-1}, 1)$ is a group and
- multiplication is compatible with the order. (It is order preserving/it distributes over join/it distributes over meet.)

Examples

- $(\mathbb{Z}, min, max, +, -, 0)$
- $(\mathbb{R}, min, max, +, -, 0)$
- $(\mathbb{C}, \vee, \wedge, +, -, 0)$, either lexicographically or coorinatewise
- The order-bijections $\mathbf{Aut}(C, \leq)$ on a chain $(C, \leq)$. For example $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

Note: special case of a residuated lattice.

Fact (The lattice reducts of) ℓ-groups are distributive. Also, the De Morgan laws hold.

Holland's embedding theorem Every ℓ-group can be embedded in $\mathbf{Aut}(\mathbf{C})$, for some chain $\mathbf{C}$.

Theorem (Weinberg) The variety of abelian ℓ-groups is generated by $\mathbb{Z}$.

The equational theory of abelian ℓ-groups is deciable via linear programing algorithms.

The variety of *representable* ℓ-groups (subdirect products of totally ordered ones) is properly between abelian and the whole variety. It is axiomatizeed by $yx \leq xyx \vee y$ and the decidability of the equational theory remains unknown.

Holland's generation theorem The variety of ℓ-groups is generated by $\mathbf{Aut}(\mathbb{R})$ (also by $\mathbf{Aut}(\mathbb{Q})$).

Theorem (Holland - McCleary) The equational theory of ℓ-groups is decidable. (Implemented online by P. Jipsen.)

If the equation is false then a finite partial description (a *diagram*) of an infinite counterexample is provided by the algorithm. If it is true, the termination of the diagram search certifies that it is false.

Fact It is enough to decide equations of the form $1 \leq g_1 \vee \cdots \vee g_n$, where $g_1, \ldots, g_n$ are group terms.

The following implications/quasiequations/inference rules hold in ℓ-groups

$$\frac{1 \le s \vee g \quad 1 \le s \vee k}{1 \le s \vee gk}\ (\textsc{mix}) \qquad \frac{1 \le s \vee gh}{1 \le s \vee g \vee h}\ (\textsc{split})$$

$$\frac{1 \le s \vee gk}{1 \le s \vee ghh^{-1}k}\ (\textsc{simp}) \qquad \frac{1 \le s \vee gk \quad s \vee nh}{1 \le s \vee gh \vee nk}\ (\textsc{com})$$

The system $\mathrm{G}\ell$ consists of the axioms and rules:

$$\frac{g \text{ gp. valid}}{1 \le s \vee g}\ (\textsc{gv}) \quad \frac{}{1 \le s \vee h \vee h^{-1}}\ (\textsc{em})$$

$$\frac{1 \le s \vee gh \quad s \vee h^{-1}k}{1 \le s \vee gk}\ (\textsc{cut}) \quad \frac{1 \le s}{1 \le s \vee t}\ (\textsc{ew})$$

Note that $(\textsc{mix})$ is an instance of $(\textsc{cut})$. Also the other three rules follow from $\mathrm{G}\ell$.

Derivable rules

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{1 \leq xxx^{-1}x^{-1} \vee yy}\,(\textsc{gv})}{1 \leq xxx^{-1} \vee yy \vee x^{-1}}\,(\textsc{split})}{1 \leq xx \vee yy \vee x^{-1}}\,(\textsc{split})}{1 \leq xx \vee yy \vee x^{-1}y^{-1}y}\,(\textsc{simp})
\qquad
\dfrac{\dfrac{\dfrac{}{1 \leq xx \vee yyy^{-1}y^{-1}}\,(\textsc{gv})}{1 \leq xx \vee yyy^{-1} \vee y^{-1}}\,(\textsc{split})}{1 \leq xx \vee yy \vee y^{-1}}\,(\textsc{split})
}{1 \leq xx \vee yy \vee x^{-1}y^{-1}}\,(\textsc{cut})
$$

For $(\textsc{split})$, $(\textsc{simp})$ and $(\textsc{com})$ we have:

$$
\dfrac{\dfrac{s \vee gh}{s \vee h \vee gh}\,(\textsc{ew}) \qquad \dfrac{}{s \vee h \vee h^{-1}}\,(\textsc{em})}{s \vee g \vee h}\,(\textsc{cut})
\qquad\qquad
\dfrac{s \vee gk \qquad \dfrac{}{s \vee k^{-1}hh^{-1}k}\,(\textsc{gv})}{s \vee ghh^{-1}k}\,(\textsc{cut})
$$

$$
\dfrac{
\dfrac{\dfrac{1 \leq s \vee n, h}{1 \leq s \vee gh \vee nh}\,(\textsc{ew})}{1 \leq s \vee gh \vee nkk^{-1}h}\,(\textsc{simp})
\qquad
\dfrac{\dfrac{1 \leq s \vee gk}{1 \leq s \vee ghh^{-1}k}\,(\textsc{simp})}{1 \leq s \vee gh \vee h^{-1}k}\,(\textsc{split})
}{1 \leq s \vee gh \vee nk}\,(\textsc{cut})
$$

A decidable system

Theorem (G. - Metcalfe) The system $\mathrm{G}\ell$ provides an axiomatization for ℓ-groups. Also, the following "resolution" rule is admissible.

$$\frac{1 \leq s \vee g \quad 1 \leq s \vee g^{-1}}{1 \leq s}\ (\textsc{res})$$

where g is not group valid.

When exploring (upward) the possible proofs of a given inequality, the choices of the subterms in $(\textsc{cut})$ and in $(\textsc{res})$ can be restricted to a finite set given by the inequality (inspired by the diagrams in Holland's proof).

This yields decidability and actually the complexity of the resulting algorithm is co-NP complete.

If the equation is true the derivation can be transformed into an equaltional-logic proof.

As a by-product, this provides an alternative proof of Holland's generation theorem without using Holland's embedding theorem.

A cut-free system

Theorem (G. - Metcalfe) The following is an alternative derivation system for ℓ-groups. Note that in the system no unexpected terms appear when reading the rules upwards.

$$\frac{}{1 \leq 1}\ (\text{EMP}) \qquad \frac{}{1 \leq xx^{-1}}\ (\text{ID}) \qquad \frac{1 \leq hg}{1 \leq gh}\ (\text{CYCLE}) \qquad \frac{1 \leq s}{1 \leq s \vee t}\ (\text{EW})$$

$$\frac{1 \leq s \vee g \quad 1 \leq s \vee h}{1 \leq s \vee gh}\ (\text{MIX}) \qquad \frac{1 \leq s \vee gk \quad 1 \leq s \vee nh}{1 \leq s \vee gh \vee nk}\ (\text{COM})$$

$$\frac{1 \leq s \vee gth \quad 1 \leq s \vee gsh}{1 \leq s \vee g(t \wedge s)h}\ (\wedge) \qquad \frac{1 \leq s \vee gth \vee gsh}{1 \leq s \vee g(t \vee s)h}\ (\vee)$$

Shortcoming Neither system allows for a good duality theory, as provided by residuated frames **(G. - Jipsen)**.

Inverse-free reducts

Fact The inverse-free reducts of ℓ-groups are necessarily distributive as lattices and multiplication distributes over both meet and join; we call such structures *totally distributive ℓ-monoids*.

Theorem (Repnitskii) The inverse-free subreducts of abelian ℓ-groups are a proper subvariety of the *commutative* totally distributive ℓ-monoids. Actually, it is not finitely based.

Theorem (Colacito - G. - Metcalfe) The inverse-free subreducts of ℓ-groups are exactly the totally distributive ℓ-monoids.

Proof-idea If an inverse-free equation fails in ℓ-groups, then it fails in $\mathbf{Aut}(\mathbf{C})$ for some chain $\mathbf{C}$. So, the (order-bijections on C in the) two sides of the equation when evaluated at some point in C produce two different values of C.

From this finite *diagram* extract/define a finite set C' of C (we take an appropriate subset of C and then duplicate elements) and endomorphisms on C' (by truncation and then extension to C') such that the two sides still evaluate at different points.

This yields a failure of the equation in the totally distributive ℓ-monoid $\mathbf{End}(\mathbf{C})$ of the endomorphisms on C'.

Inverse-free reducts of representable

Proposition (Colacito - G. - Metcalfe) The inverse-free subreducts of representatble ℓ-groups are not the whole variety of semilinear (subdirect product of chains) totally distributive ℓ-monoids.

Proof idea We define the terms

$$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8, \quad G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3,$$
$$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7, \quad G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6.$$

We show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

We also prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups. This is done by presenting a derivation in the system of **(G. - Metcalfe)** expanded by the *cycle* quasiequation ($1 \leq xy \vee z \;\Rightarrow\; 1 \leq yx \vee z$), which holds in the free representable ℓ-group.

Conjecture The inverse-free subreducts of representatble ℓ-groups do not form a finitely axiomatizable variety (over the semilinear (totally distributive) ℓ-monoids).

We should first axiomatize the variety of semilinear TDL-monoids.

Semilinear tdl-monoids

Theorem (Colacito - G. - Metcalfe) Among totally distributive ℓ-monoids the subvariety of all semilinear ones is axiomatized by the equation (esl)

$$z_1 x z_2 \wedge w_1 y w_2 \leq z_1 y z_2 \vee w_1 x w_2.$$

Theorem (G. - Horčík) A join-semilattice monoid can be embedded into the order endomorphisms $\mathbf{End}(\mathbf{C})$ of a chain $\mathbf{C}$ iff it satisfies

$$u \leq h \vee zx \;\;\&\;\; u \leq h \vee wy \Longrightarrow u \leq h \vee zy \vee wx.$$

In the lattice-ordered case this is equivalent to

$$(h \vee zx) \wedge (h \vee wy) \leq h \vee zy \vee wx.$$

In the distributive lattice-ordered case this is equivalent to

$$zx \wedge wy \leq zy \vee wx.$$

(The theorem also has versions for residuated lattices and for ℓ-groups: Holland's embeddding theorem.)

(Melier) For an monoid $\mathbf{M}$, $m \in M$ and subset I, we define

$$\frac{I}{m} = \{(x, y) \in M \times M : xmy \in I\}.$$

Also, we define a binary relation by

$$a \sim_I b \text{ iff } \tfrac{I}{a} = \tfrac{I}{b} \text{ iff for all } z, w \in M,\ zaw \in I \text{ iff } zbw \in I.$$

A *semilattice-monoid* (aka *idempotent semiring*) is a structure $\mathbf{M} = (M, \vee, \cdot, 1)$ such that $(M, \vee)$ is a join-semilattice, $(M, \cdot, 1)$ is a monoid and multiplication distributes over join on both sides.

Lemma (Melier) If I is an ideal of a semilattice-monoid, then $\sim_I$ is a congruence. If $\mathbf{M}$ is a lattice and I is $\wedge$-prime, then $\sim_I$ is compatible with meet.

In this case the quotient M/I is also a (lattice-ordered) semilattice-monoid.

Lemma (cf. G. - Horčík) The quotient M/I is a chain iff

$$z_1 x z_2 \in I \text{ and } w_1 y w_2 \in I \text{ implies } z_1 y z_2 \in I \text{ or } w_1 x w_2 \in I.$$

Lemma A semilattice monoid is semilinear iff it satisfies the implication (sl)

$$u \leq h \vee z_1 x z_2 \;\;\&\;\; u \leq h \vee w_1 y w_2 \Longrightarrow u \leq h \vee z_1 y z_2 \vee w_2 x w_2$$

Proof idea

1. relatively maximal ideals produce linear quotients (and are $\wedge$-prime in the lattice case) and that
2. we have enough relatively maximal to separate points.

Lemma If a lattice-ordered semlilattice-monoid is distributive, then (sl) is equivalent to the equation (esl): $z_1 x z_2 \wedge w_1 y w_2 \leq z_1 y z_2 \vee w_1 x w_2$.

Note that (esl) implies $ee(yx) \wedge yxe \leq ex(yx) \vee yee$, namely $yx \leq xyx \vee y$, the equation that axiomatizes representable ℓ-groups.

Starting from the system $\mathbf{DRL}$ used for distributive residuated lattices in **(G. - Jipsen)**, which does not contain transitivity/cut and is decidable, we can obtain a good derivation system $\mathbf{TDLM}$ for totally-distributive semilattice-monoids:

The system $\mathbf{DRL}$ supports the addition of equations such as distributivity of multiplication over meet: $xz \wedge xw \leq x(z \wedge w)$.

This can then be replaced by its linearized version $xz \wedge yw \leq xw \vee yz$ and then by a quasiequation $xw \leq c \;\&\; yz \leq c \Longrightarrow xz \wedge yw \leq c$. With this modification we still have completeness of the system without needing transitivity.

We can do the same for the semilinear case by transforming the equation (esl) $z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2$.

Also, we can also transform the commutativity equation $xy = yx$.

Removing inverses

Fact In abelian ℓ-groups every equation is equivalent to an inverse-free one. So, it is enough to decide the validity of inverse-free equations.

Question Is it enough to decide inverse-free equations in ℓ-groups?

Theorem (Colacito - G. - Metcalfe) The free ℓ-group satisfies:

$$u \leq h \vee cg^{-1}d \;\Leftrightarrow\; (\forall x)(gxu \leq gxh \vee gxcx \vee d).$$

There is a lose analogy with the *density rule* in proof-theory.

Corollary Every equation in ℓ-groups is equivalent to one of the form $r_0 \leq r_1 \vee \cdots \vee r_n$, where the r_i's monoid terms.

Therefore to decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in TDL-monoids.

Hybrid system: Given an ℓ-group equation we apply (upward) instances of the density rule until we obtain an inverse-free equation. Then we continue in the system $\mathbf{TDLM}$.

We can use *residuated frames* for totally distributive ℓ-monoids.

We can recover the cut-free system of **(G.-Metcalfe)**.

(Pre)orders on the free group

Fact The lattice order of any ℓ-group is the intersection of all of its total-order extensions that are *right orders* (orders compatible with right multiplication).

Fact Every total right order on a group is determined by its positive (and/or negative) cone.

Fact Total orders on the *free abelian group* on two generators are in bijective correspondence with lines through the origin with irrational slope together with (counted twice) lines through the origin with rational slope.

Theorem (Colacito - G. - Metcalfe) Let $\Sigma \cup \{t_1, \dots, , t_n\}$ be a set of group terms over the set X. The following are equivalent

1. There is no total right preorder of the *free group* over X that makes the normal closure of Σ positive and $\{t_1, \dots, , t_n\}$ strictly negative.
2. $\Sigma \models_{\mathbf{Aut}(\mathbb{Q})} e \leq t_1 \vee \cdots \vee t_n$

Corollary The following are equivalent

1. $\{t_1, \dots, , t_n\}$ does not extend to the positive cone of a right order on the free group over X.
2. $\models_{LG} 1 \leq t_1 \vee \cdots \vee t_n$

Theorem (Colacito - G. - Metcalfe) The following are equivalent

1. $\{s_1 < t_1, \dots, s_n < t_n\}$ does not extend to a right order on the *free group* over X.
2. $\models_{TDLM} y_1 s_1 \wedge \cdots \wedge y_n s_n \leq y_1 t_1 \vee \cdots \vee y_n t_n$.
3. $\{s_1 < t_1, \dots, s_n < t_n\}$ does not extend to a right order on the free monoid over X.

The variables $y_1, y_2, \dots, y_n$ are not contained in the s_i's and t_i's.