Modular Lattices Embedded into Congruence Lattices of Algebras in almost all Varieties

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Goal

$$\textbf{Con}\ \mathcal{V} = \{\textbf{Con}\ \textbf{A} : \textbf{A} \in \mathcal{V}\}$$

$$\boldsymbol{S}\,\textbf{Con}\ \mathcal{V} = \text{sublattices of these lattice}$$

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Goal: Show every modular lattice you have ever drawn (and several you haven't) is in $\boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}$, for most $\mathcal{V}$.

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$$\theta_i = \{\langle\langle a_0, a_1\rangle, \langle b_0, b_1\rangle\rangle \in A(\alpha) \times A(\alpha) : \langle a_i, b_i\rangle \in \theta\}.$$

η_0 and η_1 are the kernels of the projections (not 0_0 and 0_1).

A Useful Construction . . .

Lemma

Let α, θ and ψ be congruences on **A**. *With notation as above:*

(i) *The map $\langle a_0, a_1\rangle \mapsto \langle a_1, a_0\rangle$ defines an automorphism of* $\mathbf{A}(\alpha)$ *which interchanges θ_0 and θ_1.*

(ii) *The map $\theta \mapsto \theta_i$ is a lattice isomorphism of* **Con A** *onto the interval* $\mathbf{I}[\eta_i, 1_{\mathbf{A}(\alpha)}]$ *of* **Con** $\mathbf{A}(\alpha)$, *for $i = 0, 1$. So $(\theta \vee \psi)_i = \theta_i \vee \psi_i$ and dually.*

(iii) *If $\psi \geq \alpha$ then $\psi_0 = \psi_1$.*

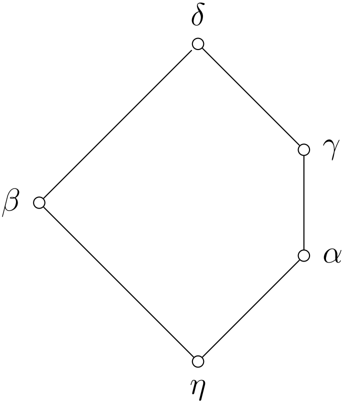
(iv) $\eta_0 \vee \eta_1 = \alpha_0$ $(= \alpha_1)$.

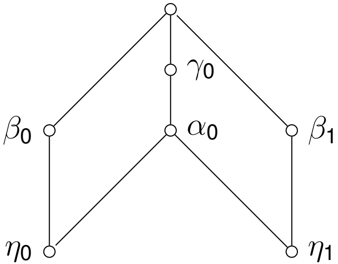
(v) *η_0 and η_1 permute.*

(vi) $(\theta_0 \wedge \theta_1) \vee \eta_0 = \theta_0$*; in fact* $\theta_0 = \eta_0 \circ (\theta_0 \wedge \theta_1) \circ \eta_0$.

Proof. Easy calculations.

A Useful Construction

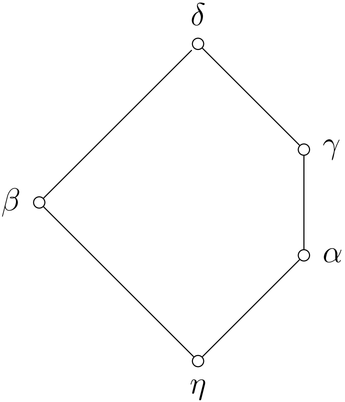




Con A

Con A(α)

A Useful Construction



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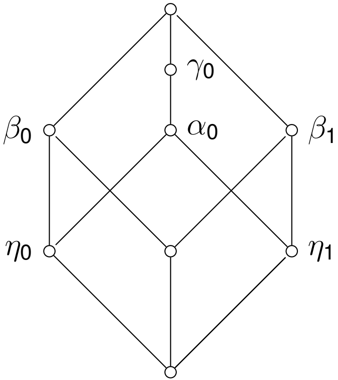

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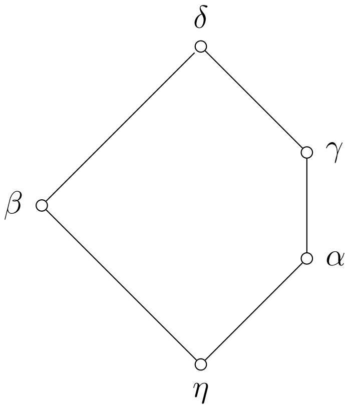

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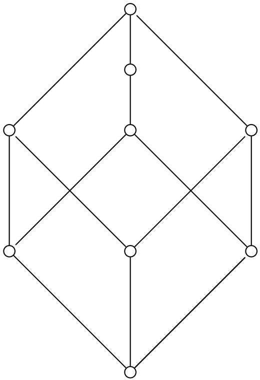

Con A$(\alpha) =$ **L**$_{14}$

So if $\mathcal{V}$ is not CM there is **A** $\in \mathcal{V}$ with **L**$_{14}$ as a sublattice.

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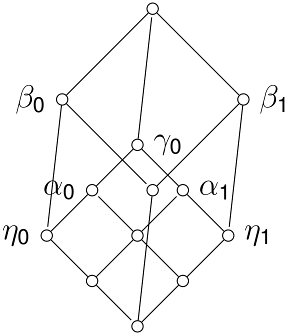

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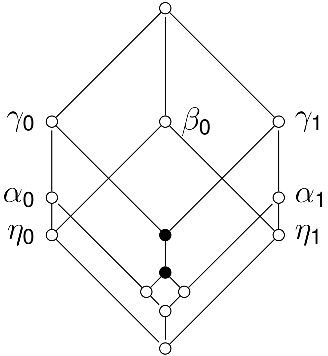

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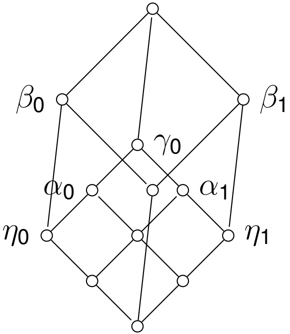

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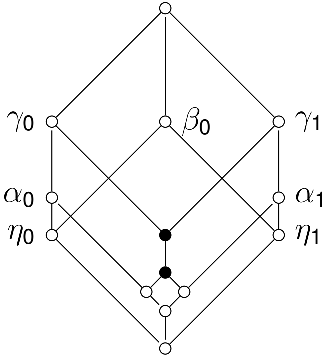

Con P(β)

Con P(β) is SI and projective so is in ***S*** **Con** $\mathcal{V}$ if it is not CM.

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- (with A. Day) $\mathcal{V}$ not CM implies $\boldsymbol{HS}\,\mathbf{Con}\ \mathcal{P} \subseteq \boldsymbol{HS}\,\mathbf{Con}\ \mathcal{V}$.
- $\mathcal{V}$ not CSD implies $\boldsymbol{HS}\,\mathbf{Con}\ \mathcal{M}_p \subseteq \boldsymbol{HS}\,\mathbf{Con}\ \mathcal{V}$, p a prime. ($\mathcal{M}_p =$ vector spaces over $\mathbb{F}_p$.)

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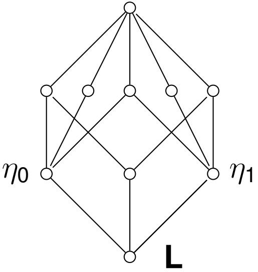
$\mathbf{M}_3$ is a cover-preserving sublattice. This is due Jónsson; the idea of the proof is on the cover of every recent AU. Applying the Useful Construction we get the following configuration which is only partial.

Modular Lattices



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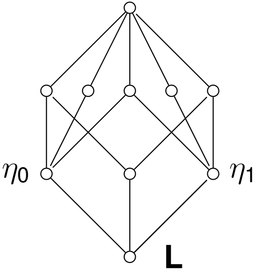

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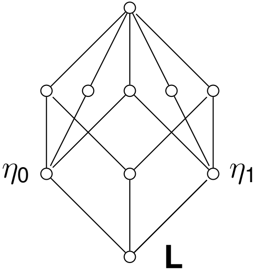

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Theorem

L *is isomorphic to lattice of subspace of a vectors space of dimension 3, or a projective plane.*

Modular Lattices

We can easily extend this to higher dimensions and conclude:

Theorem

If $\mathcal{V}$ is CM but not CD, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ *contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.*

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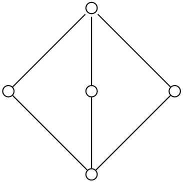
Corollary

If $\mathcal{V}$ is CM but not CD and $\mathcal{K} = \bigcap_p \mathcal{L}_p$, then

$$\boxed{\mathcal{K} \subseteq \mathbf{S}\,\mathbf{Con}\,\mathcal{V}.}$$

Modular Lattices in $\mathcal{K}$



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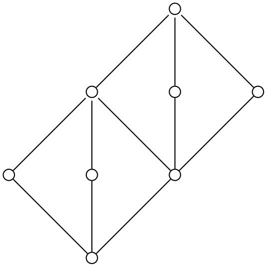

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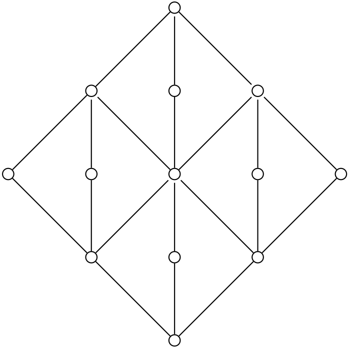

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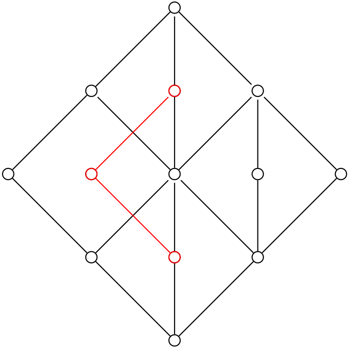

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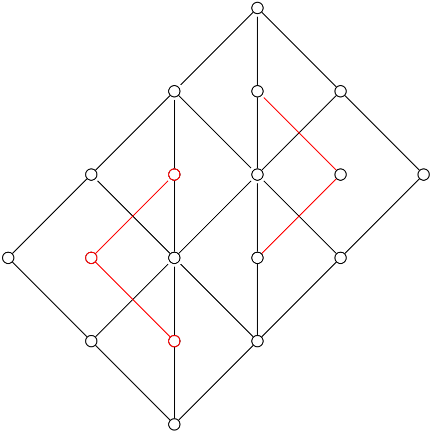

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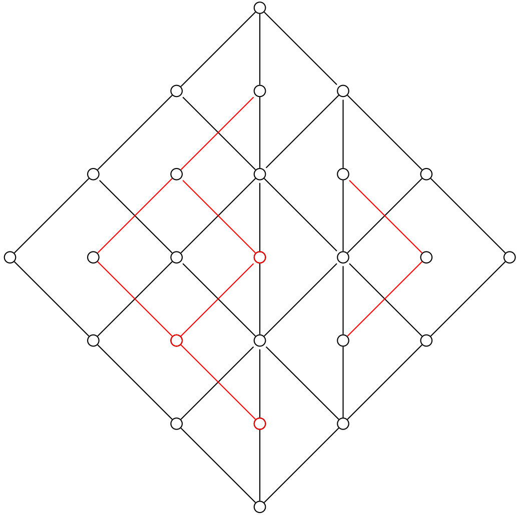

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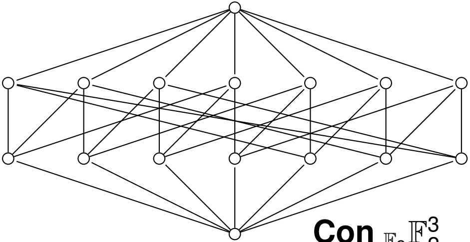

Con $_{\mathbb{F}_2}\mathbb{F}_2^3$

This lattice, the Fano plane, is not in $\mathcal{K}$.

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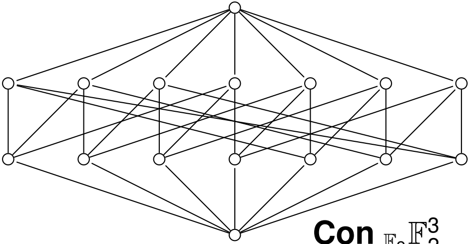

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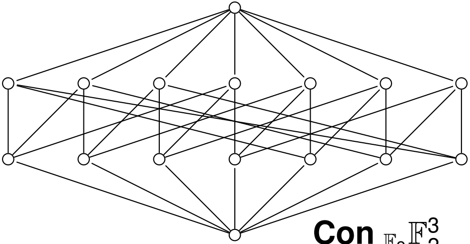

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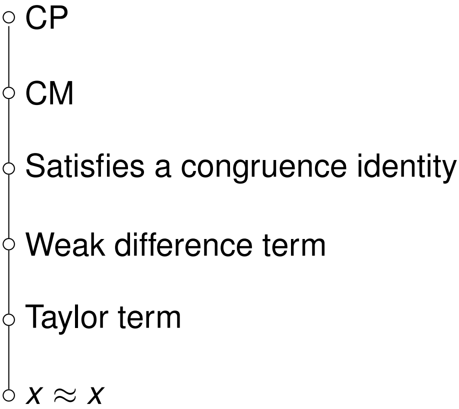
Theorem

If $\mathbf{HS}\,\mathbf{Con}\,\mathcal{V}$ *contains a nondesarguean projective plane, it satisfies no congruence identity.*

Some Mal’tsev Classes of Varieties

In Walter’s interpretability lattice:



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Theorem

If $\mathcal{V}$ has a weak difference term and has an Abelian interval, there is a p, a prime or 0, so that $\mathbf{S}\,\mathbf{Con}\,\mathcal{V}$ *contains $\mathcal{L}_p$, all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p.*

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Corollary

If $\mathcal{V}$ has a weak difference term and is not $CSD_{\wedge}$, then

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Varieties with a Taylor Term

With a weak difference term, Abelian algebras are affine.
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But we do have:

Corollary

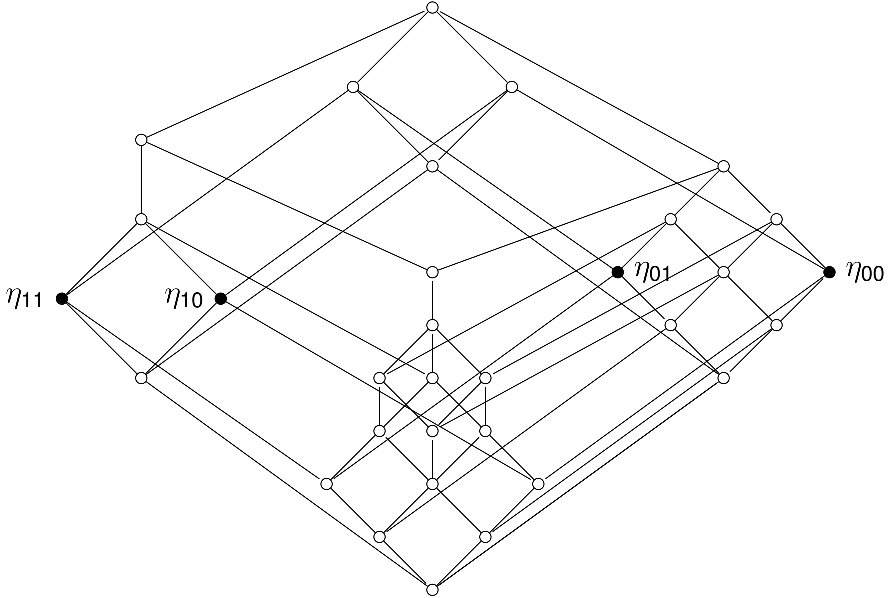
If $\mathcal{V}$ is idempotent, has a Taylor term term and is not CSD$_\wedge$, then

$$\boxed{\mathcal{K} \subseteq \boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}.}$$

Problems . . .

1. Recall $\mathcal{K} = \bigcap_p \mathcal{L}_p$, where $\mathcal{L}_p$ is all subspace lattices of all finite dimensional vector spaces over the prime field of characteristic p (including $p = 0$). Is there a good description of $\mathcal{K}$? Note it contains both simple nonplanar lattices and simple lattices that are not breadth 2.
2. Without assuming a weak difference term, if there is a proper abelian interval (somewhere in $\mathcal{V}$), can we find an algebra in $\mathcal{V}$ with congruences $\theta \succ \varphi$ satisfying $C(\theta, \theta, \varphi)$?
3. Is there an abelian algebra with a Taylor term whose congruence lattice is a descending chain such that each proper image is not abelian?
4. (See next slide) $\mathbf{Con}\,\mathbf{F}_{\mathcal{P}}(1) \in \boldsymbol{HS}\,\mathbf{Con}\,\mathcal{V}$, whenever $\mathcal{V}$ is not CM. Is it always in $\boldsymbol{S}\,\mathbf{Con}\,\mathcal{V}$?

Problems . . .



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