

ALGEBRAIC DESCRIPTION OF LIMIT MODELS IN CLASSES OF ABELIAN GROUPS

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ABSTRACT. We study limit models in the class of abelian groups with the subgroup relation and in the class of torsion-free abelian groups with the pure subgroup relation. We show:

Theorem 0.1.

- (1) If G is a limit model of cardinality λ in the class of abelian groups with the subgroup relation, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_\lambda \mathbb{Z}(p^\infty))$.
- (2) If G is a limit model of cardinality λ in the class of torsion-free abelian groups with the pure subgroup relation, then:

- If the length of the chain has uncountable cofinality, then

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_p \overline{(\oplus_\lambda \mathbb{Z}(p))}.$$

- If the length of the chain has countable cofinality, then G is not algebraically compact.

We also study the class of finitely Butler groups with the pure subgroup relation, we show that it is an AEC, Galois-stable and $(< \aleph_0)$ -tame and short.

CONTENTS

1.	Introduction	1
2.	Preliminaries	3
3.	Abelian groups	5
4.	Torsion-free abelian groups	7
5.	Finitely Butler Groups	12
	References	15

1. INTRODUCTION

Abstract elementary classes (AECs for short) were introduced in the late seventies by Shelah [Sh88] to capture the semantic structure of non-first-order theories, Shelah was interested in capturing logics like $\mathbb{L}_{\lambda^+, \omega}(\mathbf{Q})$. The setting is general enough to encompass many examples, but it still allows a development of a rich theory as witnessed by Shelah's two volume book on the subject [Sh:h] and many dozens of publications by several researchers. As a first approximation, an AEC is a class of structures with morphisms that is closed under colimits and such that every set is contained in a small model in the class.

Definition 1.1. An abstract elementary class is a pair $\mathbf{K} = (K, \leq_{\mathbf{K}})$, where:

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- (1) K is a class of τ -structures, for some fixed language $\tau = \tau(\mathbf{K})$.
- (2) $\leq_{\mathbf{K}}$ is a partial ordering on K .
- (3) $(K, \leq_{\mathbf{K}})$ respects isomorphisms: If $M \leq_{\mathbf{K}} N$ are in K and $f : N \cong N'$, then $f[M] \leq_{\mathbf{K}} N'$. In particular (taking $M = N$), K is closed under isomorphisms.
- (4) If $M \leq_{\mathbf{K}} N$, then $M \subseteq N$.
- (5) Coherence: If $M_0, M_1, M_2 \in K$ satisfy $M_0 \leq_{\mathbf{K}} M_2$, $M_1 \leq_{\mathbf{K}} M_2$, and $M_0 \subseteq M_1$, then $M_0 \leq_{\mathbf{K}} M_1$.
- (6) Tarski-Vaught axioms: Suppose δ is a limit ordinal and $\{M_i \in K : i < \delta\}$ is an increasing chain. Then:
 - (a) $M_\delta := \bigcup_{i < \delta} M_i \in K$ and $M_i \leq_{\mathbf{K}} M_\delta$ for every $i < \delta$.
 - (b) Smoothness: If there is some $N \in K$ so that for all $i < \delta$ we have $M_i \leq_{\mathbf{K}} N$, then we also have $M_\delta \leq_{\mathbf{K}} N$.
- (7) Löwenheim-Skolem-Tarski axiom: There exists a cardinal $\lambda \geq |\tau(\mathbf{K})| + \aleph_0$ such that for any $M \in K$ and $A \subseteq |M|$, there is some $M_0 \leq_{\mathbf{K}} M$ such that $A \subseteq |M_0|$ and $\|M_0\| \leq |A| + \lambda$. We write $\text{LS}(\mathbf{K})$ for the minimal such cardinal.

The main objective in the study of AECs is to develop a classification theory like the one of first-order model theory. The notions of non-forking, superstability and stability have been extended to this more general setting. The main test question is Shelah's eventual categoricity conjecture which asserts that if an AEC is categorical in *some* large cardinal then it is categorical in *all* large cardinals. Many partial results have been obtained in this direction as witnessed by for example [Sh87a], [Sh87b], [Sh99], [Sh:h], [GrVan06b], [GrVan06c], [Bon14b], [Vas17a], [Vas17b], [Vas17c], [Vas] and [ShVas].²

The notion of limit model was introduced in [KolSh96] as a substitute for saturation in the non-elementary setting (see Definition 2.9). If $\lambda > \text{LS}(\mathbf{K})$ is a regular cardinal and \mathbf{K} is an AEC with joint embedding, amalgamation and no maximal models, then: M is λ -Galois-saturated if and only if M is a (λ, λ) -limit model ([GrVas17, 2.8]).

Limit models have proven to be an important concept in tackling Shelah's eventual categoricity conjecture as witnessed by for example [ShVi99], [GrVan06a] and [Vas]. The key question has been the uniqueness of limit models of the same cardinality but with chains of different lengths. This has been studied thoroughly [ShVi99], [Van06], [GVV16], [Bon14a], [Van16], [BoVan], [ViZa16] and [Vas]. In this same line, [GrVas17] and [Vas16] showed that if a class has a monster model and is tame then uniqueness of limit models on a tail of cardinals is equivalent to being Galois-superstable³.

Despite the importance of limit models in the understanding of AECs, explicit examples have never been studied. This paper ends this by studying examples of limit models in some classes of abelian groups. The need to analyze examples is also motivated by the regular inquiry of the model theory community when presenting results on AECs. In particular, the analysis of limit models in the class of torsion-free abelian groups provides a missing example needed for [BoVan].

In this article, we study limit models in the class of abelian groups with the subgroup relation and in the class of torsion-free abelian groups with the pure subgroup relation⁴. Observe that both classes are first-order axiomatizable, but since we are studying them with a strong substructure

²For a more detailed introduction to the theory of AECs we suggest the reader to look at [Gro02], [Bal09] or [BoVas17] (this only covers tame AECs, but the AECs that we will study in this paper are all tame).

³We say that \mathbf{K} is *Galois-superstable* if there is $\mu < \beth_{(2^{\text{LS}(\mathbf{K})})^+}$ such that \mathbf{K} is λ -Galois-stable for every $\lambda \geq \mu$. Under the assumption of joint embedding, amalgamation, no maximal models and $\text{LS}(\mathbf{K})$ -tameness (which hold for all the classes studied in this paper, except perhaps the one introduced in the last section) by [GrVas17] and [Vas18] the definition of the previous line is equivalent to any other definition of Galois-superstability given in the context of AECs.

⁴Recall that H is a pure subgroup of G if for every $n \in \mathbb{N}$ it holds that $nG \cap H = nH$.

relation that is different from elementary substructure, their study is outside of the framework of first-order model theory. This freedom in choosing the strong substructure relation is a key feature of our examples and in the context of AECs has only been exploited in [BCG+] and [BET07].

The case of limit models in the class of abelian groups is simple.

Theorem 3.7. *Let $\alpha < \lambda^+$ a limit ordinal. If G is a (λ, α) -limit model in the class of abelian groups with the subgroup relation, then we have that:*

$$G \cong (\oplus_{\lambda} \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\lambda} \mathbb{Z}(p^{\infty})).$$

The case of torsion-free abelian groups (with the pure subgroup relation) is more interesting and the examination of limit models is divided into two cases. In the first one, we study limit models with chains of uncountable cofinality and by showing that they are algebraically compact we are able to give a full structure theorem. In the second one, we study limit models with chains of countable cofinality and we show that they are not algebraically compact. More precisely we obtain the following.

Theorem 4.26. *Let $\alpha < \lambda^+$ a limit ordinal. If G is a (λ, α) -limit model in the class of torsion-free abelian groups with the pure subgroup relation, then we have that:*

- (1) *If the cofinality of α is uncountable, then*

$$G \cong (\oplus_{\lambda} \mathbb{Q}) \oplus \Pi_{p \text{ prime}} \overline{(\oplus_{\lambda} \mathbb{Z}_{(p)})}.$$

- (2) *If the cofinality of α is countable, then G is not algebraically compact.*

In particular, the class does not have uniqueness of limit models for any infinite cardinal.

The paper is organized as follows. Section 2 presents necessary background. Section 3 characterizes limit models in the class of abelian groups with the subgroup relation. Section 4 studies the class of torsion-free abelian groups with the pure subgroup relation. We show that limit models of uncountable cofinality are algebraically compact (and characterize them) while those of countable cofinality are not. Section 5 studies basic properties of the class of finitely Butler groups.

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2. PRELIMINARIES

We present the basic concepts of abstract elementary classes that are used in this paper. These are further studied in [Bal09, §4 - 8] and [Gro1X, §2, §4.4]. Regarding the background on abelian groups, we assume that the reader has some familiarity with it and introduce the necessary concepts throughout the text.⁵

2.1. Basic notions. Before we introduce some concepts let us fix some notation.

Notation 2.1.

- If $M \in K$, $|M|$ is the underlying set of M .
- If λ is a cardinal, $\mathbf{K}_{\lambda} = \{M \in K : \|M\| = \lambda\}$.

⁵An excellent encyclopedic resource is [Fuc15]. We recommend the reader to keep a copy of [Fuc15] nearby since we will cite frequently from it, specially in the last section.

- Let $M, N \in K$. If we write “ $f : M \rightarrow N$ ” we assume that f is a \mathbf{K} -embedding, i.e., $f : M \cong f[M]$ and $f[M] \leq_{\mathbf{K}} N$. Observe that in particular \mathbf{K} -embeddings are always monomorphisms.

All the examples that we consider in this paper have the additional property of admitting intersections. This class of AECs was introduced in [BaSh08] and further studied in [Vas17c, §2].

Definition 2.2. An AEC admits intersections if for every $N \in K$ and $A \subseteq |N|$ there is $M_0 \leq_{\mathbf{K}} N$ such that $|M_0| = \bigcap \{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$. For $N \in K$ and $A \subseteq |N|$, we denote by $cl_{\mathbf{K}}^N(A) = \bigcap \{M \leq_{\mathbf{K}} N : A \subseteq |M|\}$, if it is clear from the context we will drop the \mathbf{K} .

Since an AEC is a semantic object, the notion of syntactic type (first-order type) does not interact well with the strong substructure relation of the AEC. Even when the AEC is axiomatizable in some extension of first-order logic, syntactic types do not behave well since equality of types does not imply the existence of \mathbf{K} -embeddings between the models mentioned in the types. For this reason Shelah introduced a notion of semantic type called Galois-type. We use the terminology of [MaVa18, 2.5].

Definition 2.3. Let \mathbf{K} be an AEC.

- (1) Let \mathbf{K}^3 be the set of triples of the form (\mathbf{b}, A, N) , where $N \in K$, $A \subseteq |N|$, and \mathbf{b} is a sequence of elements from N .
- (2) For $(\mathbf{b}_1, A_1, N_1), (\mathbf{b}_2, A_2, N_2) \in \mathbf{K}^3$, we say $(\mathbf{b}_1, A_1, N_1) E_{at} (\mathbf{b}_2, A_2, N_2)$ if $A := A_1 = A_2$, $\ell(\mathbf{b}_1) = \ell(\mathbf{b}_2)$ and there exists $f_\ell : N_\ell \xrightarrow{A} N$ such that $f_1(\mathbf{b}_1) = f_2(\mathbf{b}_2)$.
- (3) Note that E_{at} is a symmetric and reflexive relation on \mathbf{K}^3 . We let E be the transitive closure of E_{at} .
- (4) For $(\mathbf{b}, A, N) \in \mathbf{K}^3$, let $ga\text{-}tp_{\mathbf{K}}(\mathbf{b}/A; N) := [(\mathbf{b}, A, N)]_E$. We call such an equivalence class a Galois-type. Usually, \mathbf{K} will be clear from context and we will omit it.
- (5) For $ga\text{-}tp_{\mathbf{K}}(\mathbf{b}/A; N)$ and $C \subseteq A$, $ga\text{-}tp_{\mathbf{K}}(\mathbf{b}/A; N) \upharpoonright_C := [(\mathbf{b}, C, N)]_E$.

In classes that admit intersections types are easier to describe as it was shown in [Vas17c, 2.18].

Fact 2.4. Let \mathbf{K} be an AEC that admits intersections. $ga\text{-}tp(\mathbf{a}_1/A; N_1) = ga\text{-}tp(\mathbf{a}_2/A; N_2)$ if and only if there is $f : cl^{N_1}(\mathbf{a}_1 \cup A) \cong_A cl^{N_2}(\mathbf{a}_2 \cup A)$ such that $f(\mathbf{a}_1) = \mathbf{a}_2$.

The notion of Galois-stability generalizes that of a stable first-order theory. Since it will play an important role, as witness by Fact 2.11, we recall it.

Definition 2.5.

- An AEC is λ -Galois-stable if for any $M \in \mathbf{K}_\lambda$ it holds that $|gS(M)| \leq \lambda$, where $gS(M) = \{ga\text{-}tp(a/M; N) : M \leq_{\mathbf{K}} N \text{ and } a \in N\}$. Observe that $gS(M)$ denotes the 1-ary Galois-types over M .
- An AEC is Galois-stable if there is a $\lambda \geq \text{LS}(\mathbf{K})$ such that \mathbf{K} is λ -Galois-stable.

Tameness (for saturated models) appears implicitly in the work of Shelah [Sh99], but it was not until Grossberg and VanDieren isolated it in [GrVan06a] that it became a central notion in the study of AECs. Tameness was first used to prove a stability spectrum theorem in [GrVan06a] and to prove an upward categoricity transfer theorem in [GrVan06b]. For further details on tameness the reader can consult the survey by Boney and Vasey [BoVas17].

Definition 2.6. \mathbf{K} is $(< \kappa)$ -tame if for any $M \in K$ and $p \neq q \in gS(M)$, there is $A \subseteq M$ such that $|A| < \kappa$ and $p \upharpoonright_A \neq q \upharpoonright_A$.

Later, Boney isolated an analogous notion to tameness which he called type shortness in [Bon14b].

Definition 2.7. \mathbf{K} is $(< \kappa)$ -short if for any $M, N \in \mathbf{K}$, $\bar{a} \in M^\alpha$, $\bar{b} \in N^\alpha$ and $ga\text{-tp}(\bar{a}/\emptyset, M) \neq ga\text{-tp}(\bar{b}/\emptyset, N)$, there is $I \subseteq \alpha$ such that $|I| < \kappa$ and $ga\text{-tp}(\bar{a} \upharpoonright_I / \emptyset; M) \neq ga\text{-tp}(\bar{b} \upharpoonright_I / \emptyset; N)$.

2.2. Limit models. Before introducing the concept of limit model we recall the concept of universal model.

Definition 2.8. M is universal over N if and only if $N \leq_{\mathbf{K}} M$, $\|M\| = \|N\| = \lambda$ and for any $N^* \in \mathbf{K}_\lambda$ such that $N \leq_{\mathbf{K}} N^*$, there is $f : N^* \xrightarrow{N} M$.

Recall that an increasing chain $\{M_i : i < \alpha\} \subseteq \mathbf{K}$ (for α an ordinal) is a *continuous chain* if $M_i = \bigcup_{j < i} M_j$ for every $i < \alpha$ limit ordinal. With this we are ready to introduce the main concept of this paper, it was originally introduced in [KolSh96].

Definition 2.9. Let $\alpha < \lambda^+$ a limit ordinal. M is a (λ, α) -limit model over N if and only if there is $\{M_i : i < \alpha\} \subseteq \mathbf{K}_\lambda$ an increasing continuous chain such that $M_0 := N$, M_{i+1} is universal over M_i for each $i < \alpha$ and $M = \bigcup_{i < \alpha} M_i$. We say that $M \in \mathbf{K}_\lambda$ is a (λ, α) -limit model if there is $N \in \mathbf{K}_\lambda$ such that M is a (λ, α) -limit model over N . We say that $M \in \mathbf{K}_\lambda$ is a limit model if there is $\alpha < \lambda^+$ limit such that M is a (λ, α) -limit model.

Fact 2.10.

- (1) If $M \in \mathbf{K}_\lambda$ is universal over N and $M \leq_{\mathbf{K}} M^* \in \mathbf{K}_\lambda$, then M^* is universal over N .
- (2) Let \mathbf{K} be an AEC with joint embedding and amalgamation. If M is a limit model of cardinality λ , then for any $N \in \mathbf{K}_\lambda$ there is $f : N \rightarrow M$.

Proof. The first assertion is trivial so we prove the second one.

Fix $\alpha < \lambda^+$, $\{M_i : i < \alpha\}$ a witness to the fact that M is a (λ, α) -limit model and let $N \in \mathbf{K}_\lambda$. By the joint embedding property applied to M_0 and N and using the Löwenheim-Skolem-Tarski axiom there is $N^* \in \mathbf{K}_\lambda$ and $g : N \rightarrow N^*$ such that $M_0 \leq_{\mathbf{K}} N^*$. Then since M_1 is universal over M_0 , there is $h : N^* \xrightarrow{M_0} M_1$. Hence $f := h \circ g : N \rightarrow M$. \square

The following fact gives conditions for the existence of limit models.

Fact 2.11. Let \mathbf{K} be an AEC with joint embedding, amalgamation and no maximal models. If \mathbf{K} is λ -Galois-stable, then for every $N \in \mathbf{K}_\lambda$ and $\alpha < \lambda^+$ limit there is M a (λ, α) -limit model over N . Conversely, if \mathbf{K} has a limit model of cardinality λ , then \mathbf{K} is λ -Galois-stable

Proof. The forward direction is claimed in [Sh600] and proven in [GrVan06a, 2.9]. The backward direction is straightforward. \square

As mentioned in the introduction, the uniqueness of limit models of the same cardinality is a very interesting assertion. When the lengths of the cofinalities of the chains are equal, an easy back-and-forth argument gives the following.

Fact 2.12. Let \mathbf{K} be an AEC with joint embedding, amalgamation and no maximal models. If M is a (λ, α) -limit model and N is a (λ, β) -limit model such that $cf(\alpha) = cf(\beta)$, then $M \cong N$.

The question of uniqueness is intriguing when the cofinalities of the lengths of the chains are different. This question has been studied in many papers, among them [ShVi99], [Van06], [GVV16], [Bon14a], [Van16], [BoVan], [ViZa16] and [Vas].

3. ABELIAN GROUPS

In this third section, we study limit models in the class of abelian groups with the subgroup relation. Since this class was studied in great detail in [BCG+] and [BET07], the section will be short and we will cite several times.

Definition 3.1. Let $\mathbf{K}^{ab} = (K^{ab}, \leq)$ where K^{ab} is the class of abelian groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and \leq is the subgroup relation, which is the same as the substructure relation in L_{ab} .

Fact 3.2.

- (1) \mathbf{K}^{ab} is an AEC with $\text{LS}(\mathbf{K}^{ab}) = \aleph_0$.
- (2) \mathbf{K}^{ab} admits intersections.
- (3) \mathbf{K}^{ab} has joint embedding, amalgamation and no maximal models.
- (4) \mathbf{K}^{ab} is a universal class.
- (5) \mathbf{K}^{ab} is $(< \aleph_0)$ -tame and short.

Proof. (1) and (3) are shown in [BCG+, 3.3] and (2) is clear, so we show the last two assertions:

- (4) It follows from the fact that K^{ab} is axiomatizable by a set of universal first-order sentences in the language $L_{ab} = \{0\} \cup \{+, -\}$. It is fundamental that we have “-” in the language.
- (5) It follows from (4) and [Vas17c, 3.7, 3.8].

□

The following fact is implied by [BCG+, 3.4, 3.5].

Fact 3.3. Let $G \leq H$ and $a, b \in H$, the following are equivalent:

- (1) There exists $f : \text{cl}_{\mathbf{K}^{ab}}^H(G \cup \{a\}) \cong_G \text{cl}_{\mathbf{K}^{ab}}^H(G \cup \{b\})$ such that $f(a) = b$.
- (2) • $\langle a \rangle \cap G = 0 = \langle b \rangle \cap G$, or
• There are $n \in \mathbb{N}$ and $g^* \in G$ such that $na = g^* = nb$ and $ma, mb \notin G$ for all $m < n$.

In particular, \mathbf{K}^{ab} is λ -Galois-stable for every λ infinite cardinal.

Remark 3.4. Since \mathbf{K}^{ab} has joint embedding, amalgamation and no maximal models, \mathbf{K}^{ab} has limit models in every infinite cardinal by Fact 3.3 and Fact 2.11.

Recall that a group G is *divisible* if for each $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ such that $nh = g$. In the next lemma we show that limit models in \mathbf{K}^{ab} are divisible groups.

Lemma 3.5. If G is a (λ, α) -limit model, then G is a divisible group.

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α) -limit model. Let $g \in G$ and $n \in \mathbb{N}$, we want to show that $n|g$. Since $G = \bigcup_{i < \alpha} G_i$, there is $i < \alpha$ such that $g \in G_i$. Recall that every group can be embedded as a subgroup into a divisible group (see [Fuc15, §4.1.4]), so there is $D \in \mathbf{K}_\lambda$ divisible group such that $G_i \leq D$. In particular there is $d \in D$ with $nd = g$. Since G_{i+1} is universal over G_i , there is $f : D \xrightarrow{G_i} G$. Hence $nf(d) = f(g) = g$ and $f(d) \in G$. □

Using the following structure theorem for divisible groups we can characterize the limit models of \mathbf{K}^{ab} . A proof of this fact appears in [Fuc15, §4.3.1].

Fact 3.6. If G is a divisible group, then we have that:

$$G \cong (\oplus_\kappa \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\kappa_p} \mathbb{Z}(p^\infty))$$

where the cardinal numbers κ, κ_p (for all p prime number) correspond to the ranks $rk_0(G), rk_p(G)$ (for all p prime number)⁶.

From it we are able to show our first theorem.

⁶The $rk_0(G)$ is the cardinality of a maximal linearly independent subset of elements of infinite order in G and $rk_p(G)$ is the cardinality of a maximal linearly independent subset of elements of order a power of p in G . The notion of linear independence in the context of abelian groups differs slightly from that of vector spaces, the reader can consult [Fuc15, p. 91] for the definition of linear independence in this setting.

Theorem 3.7. *If G is a (λ, α) -limit model in \mathbf{K}^{ab} , then we have that:*

$$G \cong (\oplus_{\lambda} \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\lambda} \mathbb{Z}(p^{\infty})).$$

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α) -limit model. Observe that $G_0 \leq G_0 \oplus (\oplus_{\lambda} \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\lambda} \mathbb{Z}(p^{\infty}))$, therefore there is

$$f : G_0 \oplus (\oplus_{\lambda} \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\lambda} \mathbb{Z}(p^{\infty})) \xrightarrow{G_0} G.$$

In particular, $rk_0(G) = \lambda$ and $rk_p(G) = \lambda$ for all p prime, then by the structure theorem for divisible groups we have that $G \cong (\oplus_{\lambda} \mathbb{Q}) \oplus \oplus_{p \text{ prime}} (\oplus_{\lambda} \mathbb{Z}(p^{\infty}))$. \square

As a simple corollary we obtain the following.

Corollary 3.8. \mathbf{K}^{ab} has uniqueness of limit models for every infinite cardinal.

Remark 3.9. *Fact 3.3 and Fact 3.2.(3) together with [Vas18, 3.7, 11.3, 11.7] imply that \mathbf{K}^{ab} has uniqueness of limit models above $\beth_{(2^{\aleph_0})^+}$, so the result of the above corollary is only new for small cardinals.*

4. TORSION-FREE ABELIAN GROUPS

In this fourth section, we study the class of torsion-free abelian groups with the pure subgroup relation. In the first half of the section we examine basic properties of the class while in the second one we look at limit models. As we will see in this case the theory becomes more interesting.

Definition 4.1. *Let $\mathbf{K}^{tf} = (K^{tf}, \leq_p)$ where K^{tf} is the class of torsion-free abelian groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and \leq_p is the pure subgroup relation. Recall that H is a pure subgroup of G if for every $n \in \mathbb{N}$ it holds that $nG \cap H = nH$.*

4.1. Basic properties. Before analyzing the set of limit models, we obtain a few basic properties for the class of torsion-free abelian groups. As for abelian groups the basic properties of torsion-free abelian groups were studied in [BCG+] and [BET07].

Fact 4.2.

- (1) \mathbf{K}^{tf} is an AEC with $\text{LS}(\mathbf{K}^{tf}) = \aleph_0$.
- (2) \mathbf{K}^{tf} admits intersections.
- (3) \mathbf{K}^{tf} has joint embedding, amalgamation and no maximal models.

Proof. (1) and (3) are shown in [BCG+, 3.3] and [BET07] and (2) is known to hold (an argument for this is given in [Fuc15, §5.1]). \square

The following proposition characterizes the closure operator in \mathbf{K}^{tf} , since the proof is a straightforward induction we omit it.

Proposition 4.3. *If $A \subseteq H$, then $cl_{\mathbf{K}^{tf}}^H(A) = \bigcup_{n < \omega} A_n$ where:*

- $A_0 = A$.
- $A_{2k+1} = \{-h : h \in A_{2k}\} \cup \{\sum_{i=0}^n h_i : h_0, \dots, h_n \in A_{2k}, n \in \mathbb{N}\}$.
- $A_{2k+2} = \{h \in H : \text{there are } h^* \in A_{2k+1} \text{ and } n \in \mathbb{N} \text{ s.t. } nh = h^*\}$.

Recall the following definition from [Vas17c, 3.1].

Definition 4.4. \mathbf{K} is a pseudo-universal class if it admits intersections and for any $N_1, N_2 \in \mathbf{K}$ and $\bar{a}_1 \in N_1$, $\bar{a}_2 \in N_2$, if $ga\text{-tp}(\bar{a}_1/\emptyset; N_1) = ga\text{-tp}(\bar{a}_2/\emptyset; N_2)$ and $f, g : cl^{N_1}(\bar{a}_1) \cong cl^{N_2}(\bar{a}_2)$ are such that $f(\bar{a}_1) = g(\bar{a}_1) = \bar{a}_2$, then $f = g$.

The reason pseudo-universal classes will be of interest to us is due to the following statement showed in [Vas17c, 3.7].

Fact 4.5. *If \mathbf{K} is a pseudo-universal class, then \mathbf{K} is $(< \aleph_0)$ -tame and short.*

With this let us prove the following lemma.

Lemma 4.6. *\mathbf{K}^{tf} is a pseudo-universal class. In particular, \mathbf{K}^{tf} is $(< \aleph_0)$ -tame and short.*

Proof. Let $H \in \mathbf{K}^{tf}$, $\bar{a}, \bar{b} \in H$ with $\text{ga-tp}(\bar{a}/\emptyset; H) = \text{ga-tp}(\bar{b}/\emptyset; H)$ and $f, g : \text{cl}_{\mathbf{K}^{tf}}^H(\bar{a}) \cong \text{cl}_{\mathbf{K}^{tf}}^H(\bar{b})$ such that $f(\bar{a}) = g(\bar{a}) = \bar{b}$. We show by induction that $f \upharpoonright_{A_n} = g \upharpoonright_{A_n}$ for all $n < \omega$, where the A_n 's are obtained by applying Proposition 4.3 to $\text{cl}_{\mathbf{K}^{tf}}^H(\bar{a})$. The base step is the hypothesis, so we do the induction step. The odd step is straightforward, so we do the even step. Let $h \in A_{2k+2}$, by definition there is $h^* \in A_{2k+1}$ and $n \in \mathbb{N}$ such that $nh = h^*$, then since f, g are isomorphisms we have that $nf(h) = f(h^*)$ and $ng(h) = g(h^*)$. By induction hypothesis $f(h^*) = g(h^*)$, so $nf(h) = ng(h)$; using that divisors in torsion-free groups are unique, we obtain that $f(h) = g(h)$. Hence \mathbf{K}^{tf} is pseudo-universal. The fact that \mathbf{K}^{tf} is $(< \aleph_0)$ -tame and short follows from Fact 4.5. \square

In [BET07, 0.3] the following key result is obtained.

Fact 4.7. *\mathbf{K}^{tf} is λ -Galois-stable if and only if $\lambda^{\aleph_0} = \lambda$. In particular, \mathbf{K}^{tf} is a Galois-stable AEC.*

4.2. Limit models. In this subsection we classify the limit models in the class of torsion-free groups. It is clear that they are not divisible groups because if G is not divisible then G can not be a pure subgroup of a divisible group, but as we will show they are the next best thing, at least when the cofinality of the chain is uncountable. The examination of limit models will be done in two cases, we will first look at chains of uncountable cofinality and then at those of countable cofinality.

Remark 4.8. *Since \mathbf{K}^{tf} has joint embedding, amalgamation and no maximal models, \mathbf{K}^{tf} has limit models when $\lambda^{\aleph_0} = \lambda$ (and only in those cardinals) by Fact 4.7 and Fact 2.11.*

Recall the following characterization of algebraically compact groups [Fuc15, §6.1.3]. For more on algebraically compact groups the reader can consult [Fuc15, §6].

Definition 4.9. *A group G is algebraically compact if given $\mathbb{E} = \{f_i(x_{i_0}, \dots, x_{i_{n_i}}) = a_i : i < \omega\}$ a set of linear equations over G , \mathbb{E} is finitely solvable in G if and only if \mathbb{E} is solvable in G .*

Lemma 4.10. *If G is a (λ, α) -limit model and $\text{cf}(\alpha) \geq \omega_1$, then G is algebraically compact.*

Proof. Fix $\{G_\beta : \beta < \alpha\}$ a witness to the fact that G is a (λ, α) -limit model. Let $\mathbb{E} = \{f_i(x_{i_0}, \dots, x_{i_{n_i}}) = a_i : i < \omega\}$ a set of linear equations finitely solvable in G . Since $\text{cf}(\alpha) \geq \omega_1$ there is $\beta^* < \alpha$ such that $\{a_i : i < \omega\} \subseteq G_{\beta^*}$. Add new constants $\{c_i : i < \omega\}$ and consider:

$$\Sigma = \{f_i(c_{i_0}, \dots, c_{i_{n_i}}) = a_i : i < \omega\} \cup ED(G_{\beta^*}) \cup T_{tf} \cup \{\neg \exists x (nx = g) : G_{\beta^*} \models \neg \exists x (nx = g), n \in \mathbb{N}, g \in G_{\beta^*}\},$$

where T_{tf} is the first-order theory of torsion-free abelian groups and $ED(G_{\beta^*})$ is the elementary diagram of G_{β^*} .

Since \mathbb{E} is finitely solvable in G and $G_{\beta^*} \leq_p G$, it is easy to show that any finite subset of Σ is realized in G . Then by compactness and Löwenheim-Skolem-Tarski there is $H \in K_\lambda^{tf}$ such that $G_{\beta^*} \leq_p H$ (G_{β^*} is a pure subgroup by the last element in the definition of Σ) and $H \models \{f_i(c_{i_0}, \dots, c_{i_{n_i}}) = a_i : i < \omega\}$. Using the fact that G_{β^*+1} is universal over G_{β^*} , there is $f : H \xrightarrow[G_{\beta^*}]{} G_{\beta^*+1}$ and it is easy to show that $\{f(c_i^H) : i < \omega\}$ is a set of solutions to \mathbb{E} which is contained in G . \square

As a simple corollary we obtain a new proof for the following well-known assertion, the assertion without the torsion-free hypothesis appears for example in [Fuc15, §6 1.10].

Corollary 4.11. *Every torsion-free group can be embedded as a pure subgroup in a torsion-free algebraically compact group.*

Proof. Follows from the joint embedding property, Fact 2.10 and the previous lemma. \square

Before proving a theorem parallel to Theorem 3.7, we prove the following proposition. In it the group $\mathbb{Z}_{(p)}$ will play a crucial role, recall that $\mathbb{Z}_{(p)} = \{n/m : (m, p) = 1\}$.

Proposition 4.12. *If G is a (λ, α) -limit model, then $\dim_{\mathbb{F}_p}(G/pG) = \lambda$ for all p prime.⁷*

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α) -limit model. Notice that $G_0 \leq_p G_0 \oplus (\oplus_{\lambda} \mathbb{Z}_{(p)})$, then using that G_1 is universal over G_0 , there is $f : G_0 \oplus (\oplus_{\lambda} \mathbb{Z}_{(p)}) \xrightarrow{G_0} G$.

In particular, we may assume that $(\oplus_{\lambda} \mathbb{Z}_{(p)}) \leq_p G$.

Claim: $\{e_i : i < \lambda\} \subseteq (\oplus_{\lambda} \mathbb{Z}_{(p)}) \subseteq G$ satisfy that for every $g \in G$, $A \subseteq_{fin} \lambda$ and $(n_i)_{i \in A} \in \{0, \dots, p-1\}^{|A|} \setminus \{\bar{0}\}$ the following holds:

$$\Sigma_{i \in A} n_i e_i \neq pg.$$

Where each e_i is the i^{th} -element of the canonical basis.

Proof of Claim: Suppose for the sake of a contradiction that it is not the case, then there is $g \in G$, $A \subseteq_{fin} \lambda$ and $(n_i)_{i \in A} \in \{0, \dots, p-1\}^{|A|} \setminus \{\bar{0}\}$ such that

$$\Sigma_{i \in A} n_i e_i = pg.$$

Since $(\oplus_{\lambda} \mathbb{Z}_{(p)}) \leq_p G$ and $G \in K^{tf}$, we have that $g \in (\oplus_{\lambda} \mathbb{Z}_{(p)})$. Then $g = \Sigma_{i \in B} g_i$ for $B \subseteq_{fin} \lambda$ and unique $(g_i)_{i \in B} \in \mathbb{Z}_{(p)}^{|B|}$. Hence using the above equality it follows that $n_i = pg_i$ for each $i \in A$. Then p would divide the denominator of g_i for some $i \in A$, contradicting the fact that each $g_i \in \mathbb{Z}_{(p)}$, or $g = 0$, contradicting the linear independence of the e_i 's.[†]_{Claim}

From the above claim it follows that $\{e_i + pG : i < \lambda\}$ is a linearly independent set over \mathbb{F}_p . Hence $\dim_{\mathbb{F}_p}(G/pG) = \lambda$. \square

The following fact puts together the information from [EkFi72, §1] that we will need in this paper.⁸

Fact 4.13. *If G is a torsion-free algebraically compact group, then:*

$$G \cong (\oplus_{\delta} \mathbb{Q}) \oplus \Pi_p \text{ prime } \overline{(\oplus_{\beta_p} \mathbb{Z}_{(p)})}.$$

Where:

- (1) $\beta_p = \dim_{\mathbb{F}_p}(G/pG)$ for all p prime ([EkFi72, 1.7.a]).
- (2) $\delta = rk_0(G_d)$, where G_d is the maximal divisible subgroup of G ([EkFi72, 1.10]).
- (3) $\mathbb{Z}_{(p)} = \{n/m : (m, p) = 1\}$ for p prime and the overline refers to the completion⁹ (look at the discussion between [EkFi72, 1.4] and [EkFi72, 1.6]).

Lemma 4.14. *If G is a (λ, α) -limit model and G is algebraically compact, then*

$$G \cong (\oplus_{\lambda} \mathbb{Q}) \oplus \Pi_p \text{ prime } \overline{(\oplus_{\lambda} \mathbb{Z}_{(p)})}.$$

Proof. Fix $\{G_i : i < \alpha\}$ a witness to the fact that G is a (λ, α) -limit model. Since by hypothesis G is algebraically compact, by Fact 4.13 it is enough to show that $\beta_p = \lambda$ for all p prime and that $\delta = \lambda$.

By Fact 4.13.(1) and Proposition 4.12 it follows that $\beta_p = \dim_{\mathbb{F}_p}(G/pG) = \lambda$ for all p prime, so we just need to show that $\delta = \lambda$. Observe that $G_0 \leq_p G_0 \oplus (\oplus_{\lambda} \mathbb{Q})$, then there is $f :$

⁷Notice that the proposition includes the case when the cofinality of α is countable.

⁸We recommend the reader to take a look at [EkFi72, §1] or [Fuc15, §6.3].

⁹For the reader familiar with abelian group theory, this is precisely the pure-injective hull (see [Fuc15, §6.4]).

$G_0 \oplus (\oplus_\lambda \mathbb{Q}) \xrightarrow{G_0} G$, from which it follows that $rk_0(G_d) = \lambda$ since $f[(\oplus_\lambda \mathbb{Q})] \subseteq G_d$. Hence by Fact 4.13.(2), we have that $\delta = \lambda$. \square

With this we obtain our main result on limit models of uncountable cofinality.

Theorem 4.15. *If G is a (λ, α) -limit model and $cf(\alpha) \geq \omega_1$, then*

$$G \cong (\oplus_\lambda \mathbb{Q}) \oplus \Pi_p \text{ prime} \overline{(\oplus_\lambda \mathbb{Z}_{(p)})}.$$

Proof. By Lemma 4.10 G is algebraically compact. Then the result follows from Lemma 4.14. \square

The following corollary follows directly from Theorem 4.15.

Corollary 4.16. *If G is a (λ, α) -limit model and H is a (λ, β) -limit model such that $cf(\alpha), cf(\beta) \geq \omega_1$, then $G \cong H$.*

Remark 4.17. *Since \mathbf{K}^{tf} has joint embedding, amalgamation, no maximal models and is $(< \aleph_0)$ -tame, by [Vas18, 3.7] non-splitting has weak continuity and then by [Vas18, 11.3, 11.7] it follows that \mathbf{K}^{tf} has uniqueness of limit models for large λ and $cf(\alpha)$. Therefore, the result of the above corollary is only new for small cardinals.*

The next corollary follows from the above corollary doing a similar construction to [GrVas17, 2.8.(3)].

Corollary 4.18. *If G is a (λ, α) -limit model and $cf(\alpha) \geq \omega_1$, then G is λ -Galois-saturated.¹⁰*

This finishes the characterization of G when G is a (λ, α) -limit model and the cofinality of α is uncountable, we know tackle the question when the cofinality of α is countable. Regarding it, we will only have negative results, i.e., we will show that if G is a (λ, α) -limit model then G is not algebraically compact. In order to do that, we will use some deep results on AECs which appear in [GrVas17] and [Vas16]. Realize that since limit models with lengths of chains of the same cofinality are isomorphic, we only need to study (λ, ω) -limit models.

The proof will be divided into two parts. In the first we will use [GrVas17] and [Vas16] to show that for λ big (λ, ω) -limit models are not algebraically compact and in the second we will reflect the big groups into smaller cardinalities.

The following fact contains the information we will need from [GrVas17] and [Vas16]. For the readers not familiar with the theory of AECs this can be taken as a black box.

Fact 4.19. *Assume that \mathbf{K} has joint embedding, amalgamation, no maximal models, $LS(\mathbf{K}) = \aleph_0$ and is $(< \aleph_0)$ -tame. Let $\lambda \geq \beth_{(2^{\aleph_0})+\omega}$ be such that \mathbf{K} is λ -Galois-stable and there is a Galois-saturated model of cardinality λ . If every limit model of cardinality λ is Galois-saturated, then \mathbf{K} is χ -Galois-stable for every $\chi \geq \lambda$.*

Proof sketch. By [GrVas17, 3.2] \mathbf{K} does not have the \aleph_0 -order property of length $\beth_{(2^{\aleph_0})+}$. Then by [GrVas17, 3.18] \mathbf{K} has no long splitting chains in λ . Since \mathbf{K} has no long splitting chains in λ , is λ -Galois-stable and is $(< \aleph_0)$ -tame by [Vas16, 5.6] we can conclude that \mathbf{K} is χ -Galois-stable for every $\chi \geq \lambda$. \square

Lemma 4.20. *Let $\lambda \geq \beth_{(2^{\aleph_0})+\omega}$. If G is a (λ, ω) -limit model, then G is not algebraically compact.*

¹⁰Recall that G is λ -Galois-saturated if for every $H \leq_{\mathbf{K}} G$ and $p \in \text{gS}(H)$ such that $\|H\| < \lambda$, p is realized in G . G is Galois-saturated if it is $\|G\|$ -Galois-saturated.

Proof. Since G is a (λ, ω) -limit model, it follows that \mathbf{K}^{tf} is λ -Galois-stable by Fact 2.11.

Assume for the sake of contradiction that G is algebraically compact, then by Lemma 4.14 $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \prod_p \text{prime}(\oplus_\lambda \mathbb{Z}_{(p)})$. Then by Theorem 4.15 \mathbf{K}^{tf} has uniqueness of limit models of cardinality λ . Hence every limit model of cardinality λ is Galois-saturated by [GrVas17, 2.8.(3)].

By Fact 4.2 and Lemma 4.6 \mathbf{K}^{tf} has joint embedding, amalgamation, no maximal models, $\text{LS}(\mathbf{K}^{tf}) = \aleph_0$ and is $(< \aleph_0)$ -tame. Then by Fact 4.19 \mathbf{K}^{tf} is χ -Galois-stable for every $\chi \geq \lambda$. But this contradicts Fact 4.7, since there is $\chi \geq \lambda$ such that $\chi^{\aleph_0} \neq \chi$. \square

Lemma 4.21. *Let $\lambda < \beth_{(2^{\aleph_0})+\omega}$. If G is a (λ, ω) -limit model, then G is not algebraically compact.*

Proof. Since G is a (λ, ω) -limit model, it follows that \mathbf{K}^{tf} is λ -Galois-stable by Fact 2.11.

Let $\mu \geq \beth_{(2^{\aleph_0})+\omega}$ such that $\mu^{\aleph_0} = \mu$, by Fact 4.7 \mathbf{K}^{tf} is μ -Galois-stable. Let G^* a (μ, ω) -limit model witnessed by $\{G_i^* : i < \omega\}$. By Lemma 4.20 G^* is not algebraically compact, so there is $\mathbb{E} = \{f_k(x_{k_0}, \dots, x_{k_{n_k}}) = a_k : k < \omega\}$ a set of linear equations finitely solvable in G^* but not solvable in G^* .

We build $\{r_i : i < \omega\} \subseteq \mathbb{N}$, $\{S_i : i < \omega\}$ and $\{H_i : i < \omega\}$ by induction such that:

- (1) $\{r_i : i < \omega\}$ is strictly increasing.
- (2) $a_i \in H_i$.
- (3) $S_i \subseteq H_i$ and S_i is a finite set.
- (4) S_i has a solution to $\{f_k(x_{k_0}, \dots, x_{k_{n_k}}) = a_k : k \leq i\}$.
- (5) $H_i \leq_p G_{r_i}^*$.
- (6) $H_i \in \mathbf{K}_\lambda^{tf}$.
- (7) H_{i+1} is universal over H_i .

Before we do the construction, let us show that this is enough. Let $H_\omega := \bigcup_{i < \omega} H_i$, by (6) and (7) it follows that H_ω is a (λ, ω) -limit model. Since limit models of the same cofinality are isomorphic by Fact 2.12, it follows that $H_\omega \cong G$. So it is enough to show that H_ω is not algebraically compact. Assume for the sake of contradiction that H_ω is algebraically compact. Since $\mathbb{E} = \{f_k(x_{k_1}, \dots, x_{k_{n_k}}) = a_k : k < \omega\}$ is finitely solvable in H_ω by (4), it follows that there is $\mathbf{a} \in H_\omega^\omega$ a solution for \mathbb{E} . But this contradicts the fact that \mathbb{E} is not solvable in G^* , since $H_\omega \leq_p G^*$ by (5). Therefore, H_ω is not algebraically compact.

Now let us do the construction.

Base Let $\{b_0, \dots, b_l\} \subseteq G^*$ a solution to $f_0(x_{0_0}, \dots, x_{0_{n_0}}) = a_0$, this exists by finite solvability of \mathbb{E} in G^* , and $r < \omega$ such that $\{b_0, \dots, b_l, a_0\} \subseteq G_r^*$. Let $r_0 := r$, $S_0 := \{b_0, \dots, b_l\}$ and applying Löwenheim-Skolem-Tarski axiom to $\{b_0, \dots, b_l, a_0\}$ in $G_{r_0}^*$ we get $H_0 \in \mathbf{K}_\lambda^{tf}$ such that $H_0 \leq_p G_{r_0}^*$ and $\{b_0, \dots, b_l, a_0\} \subseteq H_0$. It is easy to see that this works.

Induction step By construction there are $r_i \in \mathbb{N}$ and $H_i \leq_p G_{r_i}^*$. Since \mathbf{K}^{tf} is λ -Galois-stable we can build $H \in \mathbf{K}_\lambda^{tf}$ such that H is universal over H_i by Fact 2.11. Using that $H_i \leq_p G_{r_i}^*$, the amalgamation property and that $G_{r_{i+1}}^*$ universal over $G_{r_i}^*$, there is $f : H \xrightarrow{H_i} G_{r_{i+1}}^*$.

Let $\{b_0, \dots, b_l\} \subseteq G^*$ a solution to $\{f_k(x_{k_0}, \dots, x_{k_{n_k}}) = a_k : k \leq i+1\}$ and take $r \geq r_i + 1$ such that $\{b_0, \dots, b_l, a_{i+1}\} \subseteq G_r^*$. Let $r_{i+1} := r$, $S_{i+1} := \{b_0, \dots, b_l\}$ and applying Löwenheim-Skolem-Tarski axiom to $f[H] \cup \{b_0, \dots, b_l, a_{i+1}\}$ in $G_{r_{i+1}}^*$ we get $H_{i+1} \in \mathbf{K}_\lambda^{tf}$ such that $H_{i+1} \leq_p G_{r_{i+1}}^*$ and $f[H] \cup \{b_0, \dots, b_l, a_{i+1}\} \subseteq H_{i+1}$. Using that $H_i \leq_p f[H] \leq_p H_{i+1}$ and that $f[H]$ is universal over H_i , it is easy to show that (1) through (7) hold. \square

Putting together the last two lemmas we obtain the following.

Theorem 4.22. *If G is a (λ, ω) -limit model, then G is not algebraically compact.*

Proof. If $\lambda \geq \beth_{(2^{\aleph_0})^{++\omega}}$ it follows from Lemma 4.20 and if $\lambda < \beth_{(2^{\aleph_0})^{++\omega}}$ it follows from Lemma 4.21. \square

Remark 4.23. *After discussing Theorem 4.22 with Sebastien Vasey, he realized that by applying [Vas18, 4.12] instead of [GrVas17, 3.18] one could prove Theorem 4.22 without dividing the proof into cases. The proof using [Vas18, 4.12] is similar to that of Lemma 4.20. We decided to keep our original argument since the proof presented here shows how to transfer the failure of being algebraically compact and since we believe that showing that there are cofinally many (λ, ω) -limit models that are not algebraically compact is provable using only group theoretic methods.*

Since (λ, ω) -limit models are not algebraically compact we ask:

Question 4.24. *Is there a natural class of groups that contain the (λ, ω) -limit models?*

Regarding the structure of (λ, ω) -limit models, using the fact that every group is a direct sum of a divisible group and a reduced group¹¹ (see [Fuc15, §4.2.5]), it is straightforward to show that if G is a (λ, ω) -limit model, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus G_r$ where $G_r \cong G/G_d$, G_d is the maximal divisible subgroup of G and G_r is reduced. So it is natural to ask the following.

Question 4.25. *Is there a structure theorem for (λ, ω) -limit models similar to that of Theorem 4.15?*

Let us conclude with the main theorem of this section.

Theorem 4.26. *If G is a (λ, α) -limit model in \mathbf{K}^{tf} , then we have that:*

- (1) *If the cofinality of α is uncountable, then $G \cong (\oplus_\lambda \mathbb{Q}) \oplus \prod_{p \text{ prime}} \overline{(\oplus_\lambda \mathbb{Z}_{(p)})}$.*
- (2) *If the cofinality of α is countable, then G is not algebraically compact.*

In particular, \mathbf{K}^{tf} does not have uniqueness of limit models for any infinite cardinal.

Proof. The first part is Theorem 4.15 and the second one is Theorem 4.22. The “in particular” follows from the fact that limit models with chains of uncountable cofinality are algebraically compact by (1), while those with chains of countable cofinality are not algebraically compact by (2). \square

5. FINITELY BUTLER GROUPS

In this last section, we look at some basic properties of the class of finitely Butler groups. The results in this section are weaker than those of the previous two sections and in some sense incomplete, but we decided to present them since we see this section as a stepping stone and moreover finitely Butler groups had never been isolated as an AEC.

Butler groups were introduced by Butler in [But65], while finitely Butler groups were first studied in [BiSa83] and given a name in [FuVi90]. We follow the exposition of [Fuc15, §14] and recommend the reader to consult it for further details.

Definition 5.1. *A torsion-free group G of finite rank¹² is a Butler group if G is a pure subgroup of a finite rank completely decomposable group. Recall that a torsion-free group is completely decomposable if and only if it is the direct sum of groups of rank one.*

Definition 5.2. *A torsion-free group G is a finitely Butler group (B_0 -group) if every pure subgroup of finite rank of G is a Butler group.*

Let us introduce the class we will study.

¹¹Recall that a group H is reduced if its only divisible subgroup is 0.

¹²Given G a torsion-free group, the rank of G is $rk_0(G)$ (see footnote 6 for the definition).

Definition 5.3. Let $\mathbf{K}^{B_0} = (K^{B_0}, \leq_p)$ where K^{B_0} is the class of finitely Butler groups in the language $L_{ab} = \{0\} \cup \{+, -\}$ and \leq_p is the pure subgroup relation.

Remark 5.4. Notice that if $G \in K^{B_0}$ and $H \leq_p G$, then $H \in K^{B_0}$.

Our first assertion is that indeed \mathbf{K}^{B_0} is an AEC.

Lemma 5.5. $\mathbf{K}^{B_0} = (K^{B_0}, \leq_p)$ is an AEC with $\text{LS}(\mathbf{K}^{B_0}) = \aleph_0$ that admits intersections.

Proof. From the closure under pure subgroups and the fact that \mathbf{K}^{tf} is an AEC, it follows that \mathbf{K}^{B_0} satisfies all the axioms of an AEC except the first Tarski-Vaught axiom. We show that it holds.¹³

Let $\{G_i : i < \delta\}$ such that $G_i \leq_p G_j$ for all $i < j$ and $G = \bigcup_{i < \delta} G_i$. It is clear that $G_i \leq_p G$ for all $i < j$, so we only need to show that $G \in K^{B_0}$, so let $H \leq_p G$ of finite rank.

Take X a finite maximal linearly independent subset of H , it exists because H has finite rank. Since X is finite, there is $i < \delta$ such that $X \subseteq G_i$. Since X is maximal linearly independent $H \subseteq \text{span}_{\mathbb{Q}}(X)$. Then using that $G_i \leq_p G$ and G_i is torsion-free, it follows that $H \leq_p G_i$. Therefore, since $G_i \in K^{B_0}$, we conclude that H is a Butler group.

Moreover, the class admits intersections because \mathbf{K}^{tf} admits intersections and the closure of \mathbf{K}^{B_0} under pure subgroups. \square

Fact 5.6. \mathbf{K}^{B_0} has joint embedding and no maximal models.

Proof. By [Fuc15, §14.5.(B)] \mathbf{K}^{B_0} is closed under direct sums so the result follows. \square

Regarding the amalgamation property, we are only able to provide the following partial solution. We actually think that the amalgamation property might not hold for the class.

Lemma 5.7. If $G \in K^{B_0}$ and G is divisible, then G is an amalgamation base, i.e., if $G \leq_p H_i \in K^{B_0}$ for $i \in \{1, 2\}$, then there are $H \in K^{B_0}$ and $f_i : H_i \rightarrow H$ for $i \in \{1, 2\}$ such that $f_1 \upharpoonright_G = f_2 \upharpoonright_G$.

Proof. Let $G \leq_p H_i$ for $i \in \{1, 2\}$. Let $H := H_1 \oplus H_2/G^*$ where $G^* := \{(g, -g) : g \in G\}$, $f_1 : H_1 \rightarrow H$ be $f_1(h) = (h, 0) + G^*$ and $f_2 : H_2 \rightarrow H$ be $f_2(h) = (0, h) + G^*$. In [BCG+, 3.27] it is shown that $H \in K^{tf}$, f_1, f_2 are pure embeddings and $f_1 \upharpoonright_G = f_2 \upharpoonright_G$. So we only need to show that $H \in K^{B_0}$.

Let $E \subseteq H_1 \oplus H_2$ such that $E/G^* \leq_p H_1 \oplus H_2/G^*$ and E/G^* has rank n . Take $\{e_i + G^* : i < n\}$ a maximal linearly independent subset of E/G^* .

Observe that $E \leq_p H_1 \oplus H_2$, because $G^* \leq_p H_1 \oplus H_2$ and $E/G^* \leq_p H_1 \oplus H_2/G^*$. Moreover, $cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\}) \leq_p H_1 \oplus H_2$, $cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$ has finite rank and $H_1 \oplus H_2 \in K^{B_0}$ (see [Fuc15, §14.5.(B)]), so it follows that $cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$ is a Butler group (where the closure is the one described in Proposition 4.3 by Remark 5.4).

Claim: $E = G^* + cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$.

Proof of Claim: Let $e \in E$, since $\{e_i + G^* : i < n\}$ is maximal linearly independent $e + G^* \in \text{span}_{\mathbb{Q}}(\{e_i + G^* : i < n\})$, then there are $\{m, k_0, \dots, k_{n-1}\} \subseteq \mathbb{N}$ and $g_0^* \in G^*$ such that:

$$me = \sum_{i=0}^{n-1} k_i e_i + g_0^*.$$

Since G is divisible, G^* is divisible so there is $g_1^* \in G^*$ such that $mg_1^* = g_0^*$. Then $m(e - g_1^*) = \sum_{i=0}^{n-1} k_i e_i$, thus $e - g_1^* \in cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$. Hence $e \in G^* + cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$. \dagger Claim

Then $E/G^* \cong G^* + cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})/G^* \cong cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})/cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\}) \cap G^*$. By the fact that torsion-free epimorphic images of Butler groups are Butler groups (see [Fuc15, §14.1.6]) and that $cl_{\mathbf{K}^{B_0}}^E(\{e_0, \dots, e_{n-1}\})$ is a Butler group, we conclude that E/G^* is a Butler group. Hence $H \in K^{B_0}$. \square

¹³This is exercise [Fuc15, §14.4.1].

The next proposition is straightforward, but we include it because of its strong consequences.

Proposition 5.8. *If $G, H \in K^{B_0}$, $\mathbf{a} \in G$, $\mathbf{b} \in H$ and $A \subseteq G, H$, then $\text{ga-tp}_{\mathbf{K}^{B_0}}(\mathbf{a}/A; G) = \text{ga-tp}_{\mathbf{K}^{B_0}}(\mathbf{b}/A; H)$ if and only if $\text{ga-tp}_{\mathbf{K}^{tf}}(\mathbf{a}/A; G) = \text{ga-tp}_{\mathbf{K}^{tf}}(\mathbf{b}/A; H)$.*

Proof. Since \mathbf{K}^{B_0} is closed under pure subgroups by Remark 5.4, using the minimality of the closures, it is easy to show that for all $H' \in K^{B_0}$ and $B \subseteq H'$ it holds that $cl_{\mathbf{K}^{B_0}}^{H'}(B) = cl_{\mathbf{K}^{tf}}^{H'}(B)$. Then using that \mathbf{K}^{B_0} and \mathbf{K}^{tf} admit intersections and Fact 2.4 the result follows. \square

Corollary 5.9.

- \mathbf{K}^{B_0} is $(< \aleph_0)$ -tame and short.
- If $\lambda = \lambda^{\aleph_0}$, then \mathbf{K}^{B_0} is λ -Galois-stable. In particular, \mathbf{K}^{B_0} is a Galois-stable AEC.

Proof. The proof follows directly from Proposition 5.8 and the fact that \mathbf{K}^{tf} satisfies both of the properties we are trying to show. \square

Question 5.10. *Do we have as in \mathbf{K}^{tf} that: if \mathbf{K}^{B_0} is λ -Galois-stable, then $\lambda = \lambda^{\aleph_0}$?*

We were unable to answer the above question, but we have a partial solution (see Lemma 5.12). In order to present it, we will need some results from [Fuc15, §12.1] and the following definitions.

Definition 5.11. *Let G be a torsion-free abelian group and $a \in G$:*

- Given a prime p the p -height of a (denoted by $h_p(a)$) is the maximum $n \in \mathbb{N}$ such that $p^n | a$ or ∞ if the maximum does not exist.
- The characteristic of a is $\chi_G(a) = (h_{p_n}(a))_{n < \omega}$ where $\{p_n : n < \omega\}$ is an increasing enumeration of the prime numbers.
- Given $\eta, \nu \in (\mathbb{N} \cup \{\infty\})^\omega$ we define the equivalence relation \sim as $\eta \sim \nu$ if and only if η and ν differ on finitely many natural numbers and when they differ they are both finite. A type \mathbf{t} is an element of $(\mathbb{N} \cup \{\infty\})^\omega / \sim$ and the type of a is $\mathbf{t}_G(a) = \chi_G(a) / \sim$.
- We say that G has type \mathbf{t} , if for every $b \neq 0 \in G$ it holds that $\mathbf{t} = \mathbf{t}_G(b)$.

The proof of the following lemma uses similar ideas to those of [KojSh95, 3.7].

Lemma 5.12. *If $\lambda < 2^{\aleph_0}$, then \mathbf{K}^{B_0} is not λ -Galois-stable.*

Proof. Let $G \in \mathbf{K}_\lambda^{B_0}$ and $\{\mathbf{t}_\eta : \eta \in 2^\omega\}$ an enumeration of all the types (in the sense of the previous definition). For each $\eta \in 2^\omega$, let G_η a group of rank one with type \mathbf{t}_η , it exists by [Fuc15, §12.1.1]. Let $H = G \oplus (\oplus_{\eta \in 2^\omega} G_\eta)$. Since \mathbf{K}^{B_0} is closed under direct sums (see [Fuc15, §14.5.(B)]) and rank one groups are in \mathbf{K}^{B_0} , because they are completely decomposable, we have that $H \in K^{B_0}$.

For each $\eta \in 2^\omega$ take $a_\eta \in G_\eta$ with $a_\eta \neq 0$ and let $p_\eta := \text{ga-tp}(a_\eta/G; H)$. We show that all the Galois-types in the set $\{p_\eta : \eta \in 2^\omega\}$ are different.

Claim: If $\eta \neq \nu \in 2^\omega$, then $p_\eta \neq p_\nu$.

Proof of Claim: Suppose for the sake of contradiction that $\text{ga-tp}(a_\eta/G; H) = \text{ga-tp}(a_\nu/G; H)$, then by Fact 2.4 there is $f : cl_{\mathbf{K}^{B_0}}^H(\{a_\eta\} \cup G) \cong_G cl_{\mathbf{K}^{B_0}}^H(\{a_\nu\} \cup G)$ with $f(a_\eta) = a_\nu$. Then since the closures give rise to pure subgroups of H we have that $\chi_H(a_\eta) = \chi_H(a_\nu)$, so $\mathbf{t}_H(a_\eta) = \mathbf{t}_H(a_\nu)$. This contradicts the fact that $\mathbf{t}_H(a_\eta) = \mathbf{t}_{G_\eta}(a_\eta) = \mathbf{t}_\eta \neq \mathbf{t}_\nu = \mathbf{t}_{G_\nu}(a_\nu) = \mathbf{t}_H(a_\nu)$, the first and last equality follow from the fact that $G_\eta, G_\nu \leq_p H$. \dagger_{Claim}

Therefore, $|\text{gS}(G)| \geq 2^{\aleph_0}$. Since $\lambda < 2^{\aleph_0}$, \mathbf{K} is not λ -Galois-stable. \square

As we mentioned in the introduction we are interested in limit models, therefore we ask the following:

Question 5.13. *Do limit models exist in \mathbf{K}^{B_0} ? If they exist, what is their structure?*

Regarding the first part of the question, realize that if \mathbf{K}^{B_0} has the amalgamation property, then by Corollary 5.9 and Fact 2.11 limit models would exist. As for the second part, even if they existed the techniques to characterize them would have to be different from the ones presented in section four since finitely Butler groups do not seem to be first-order axiomatizable.

Besides the function of this article as a pool of examples of limit models in the context of AECs. We believe that the study of limit models (in different classes of groups) as a classes of infinite rank groups could be an interesting area of research on its own. We think this is possible since limit model are tame enough to be analyzable, but their theory is nontrivial as showcased in this article. A good place to look for new classes of limit models is [BET07].

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