Math 6230: Spring 2020

1 The Rank Theorem

Theorem 1.1. Let $M, N$ be smooth manifolds such that $\dim M = m, \dim N = n$, and let $F : M \to N$ be a smooth map with constant rank $r$. For each $p \in U$, there exists a chart $(U, \varphi)$ centered at $p$, and a chart $(V, \psi)$ centered at $F(p)$, with $F(U) \subset V$ such that

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = \psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

The proof of the theorem is based on the following lemmas and a result in the Euclidean setting.

Lemma 1.2 (Prop. B.25 in Lee). Suppose $A$ is an $m \times n$ matrix. Then $\text{rank} A \geq k$ if and only if some $k \times k$ submatrix of $A$ is nonsingular.

Lemma 1.3 (Prob. B.28 in Lee). Let $X$ be an $(m+k) \times (m+k)$ block matrix, given by

$$[A \ B] [0 \ C],$$

where $A$ is $m \times m$, $B$ is $m \times k$, and $C$ is a $k \times k$ matrix. Then

$$\det X = \det A \det C.$$

Lemma 1.4. Let $F : M \to N$ be a smooth map with rank $k$, and let $\phi : M \to M$ be a diffeomorphism. Then $F \circ \phi$ has rank $k$.

Proof. Follows from Exercise B.22 in Lee.

Theorem 1.5 (Euclidean Rank Theorem). Suppose $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n, U, V$ are open. Let $F : U \to V,$

be smooth and have a constant rank $r$. For all $p \in U$, there exists a chart $(U_0, \varphi)$ centered at $p$ ($\varphi(p) = 0$), and a chart $(V_0, \psi)$ centered at $F(p)$, with $U_0 \subset U$ and $F(U_0) \subset V_0 \subset V$ such that

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = \psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Outline of the proof of Theorem 1.5

• Step 1: Set-up
• Step 2: Defining $\varphi$ and applying the inverse function theorem.
• Step 3: Examining $\varphi^{-1}$ and $F \circ \varphi^{-1}$.
• Step 4: Finding $\psi$. 

1