Math 6230: Spring 2020

1 The Rank Theorem

Theorem 1.1. *Let M, N be smooth manifolds such that $\dim M = m, \dim N = n$, and let $F : M \to N$ be a smooth map with constant rank r. For each $p \in U$, there exists a chart (U, φ) centered at p, and a chart (V, ψ) centered at $F(p)$, with $F(U) \subset V$ such that*

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = \psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

The proof of the theorem is based on the following lemmas and a result in the Euclidean setting.

Lemma 1.2 (Prop. B.25 in Lee)**.** *Suppose A is an $m \times n$ matrix. Then rank$A \geq k$ if and only if some $k \times k$ submatrix of A is nonsingular.*

Lemma 1.3 (Prob. B.28 in Lee)**.** *Let X be an $(m+k) \times (m+k)$ block matrix, given by*

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where A is $m \times m$, B is $m \times k$, and C is a $k \times k$ matrix. Then

$$\det X = \det A \det C.$$

Lemma 1.4. *Let $F : M \to N$ be a smooth math with rank k, and let $\phi : M \to M$ be a diffeomorphism. Then $F \circ \phi$ has rank k.*

Proof. Follows from Exercise B.22 in Lee. □

Theorem 1.5 (Euclidean Rank Theorem)**.** *Suppose $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$, U, V are open. Let*

$$F : U \to V,$$

be smooth and have a constant rank r. For all $p \in U$, there exists a chart (U_0, φ) centered at p ($\varphi(p) = 0$), and a chart (V_0, ψ) centered at $F(p)$, with $U_0 \subset U$ and $F(U_0) \subset V_0 \subset V$ such that

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = \psi \circ F \circ \varphi^{-1}(x^1, \ldots, x^r, x^{r+1}, \ldots x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

Outline of the proof of Theorem 1.5

- Step 1: Set-up
- Step 2: Defining φ and applying the inverse function theorem.
- Step 3: Examining φ^{-1} and $F \circ \varphi^{-1}$.
- Step 4: Finding ψ.