Introduction to measure theory and construction of the Lebesque measure

Intro to measure theory: based on baby Rudin
Construction of the Lebesgue measure: based on Jones (Lebesgue Integration on Euclidean Space)

Analysis 2, Fall 2019
University of Colorado Boulder

Outline

Introduction

Intro to measure theory
- ring of sets
- set functions on $\mathcal{R}$

Construction of Lebesgue measure based on Jones
- Steps 0-1: empty set, intervals
- Steps 2-4: special polygons, open and compact sets
- Outer & Inner measures: Step 5
- Step 6
- Properties of Lebesgue measure

Introduction

- Ultimate goal is to learn *Lebesgue integration*.
- Lebesgue integration uses the concept of a measure.
- Before we define Lebesgue integration, we define one concrete measure, which is the *Lebesgue measure* for sets in $\mathbb{R}^n$.
- Then, when we start talking about the Lebesgue integration, we can think about abstract measures or have this concrete example of the Lebesgue measure in mind.
- The proofs omitted in lecture will be either left as homework, exercise or you will not be responsible for knowing the proof.

Introduction

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- Lebesgue integration uses the concept of a measure.
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Intro to measure theory

- Not every set has a well defined Lebesgue measure, so when we define Lebesgue measure we also talk about a *family of sets* for which the measure is well defined.
- In fact, this idea shows up in abstract measure theory: family of sets for which the abstract measure is defined.
- So we will discuss:
  - families of sets
  - what is a definition of a measure

Intro to measure theory: ring of sets

Let A and B be two sets. Recall

$$A - B = \{x : x \in A, x \notin B\}.$$

Note B does not have to be contained in A to consider $A - B$.

Definition

A family $\mathcal{R}$ of sets is a *ring* if and only if $A, B \in \mathcal{R}$, then

$$A \cup B \in \mathcal{R}, \quad \text{and} \quad A - B \in \mathcal{R}.$$

Theorem

Let $\mathcal{R}$ be a ring, and $A, B \in \mathcal{R}$, then

$$A \cap B \in \mathcal{R}.$$

Proof.

Obvious once we observe that $A \cap B$ can be written as $A - (A - B)$. □

Intro to measure theory: σ-rings

Definition

A ring $\mathcal{R}$ is a $\sigma-$ring if and only if $\cup_{i=1}^{\infty} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}$ for all i.

(So a σ-ring is a ring that is closed under countable unions.)

Theorem

Let $\mathcal{R}$ be a σ-ring, and A_i be a collection of sets such that $A_i \in \mathcal{R}$ for all i, then

$$\cap_{i=1}^{\infty} A_i \in \mathcal{R}.$$

Proof.

Exercise. □

Remark: Eventually we will discuss a σ-ring of Lebesgue measurable sets. Right now we are just collecting definitions, and keeping everything abstract, so if we wanted to, we could define other measures besides the Lebesgue measure.

Set functions on $\mathcal{R}$ (*secretly:* within those are candidates for measures)

Definition

A function $\phi : \mathcal{R} \to [-\infty, \infty]$ is called a set function on $\mathcal{R}$.

A set function can be *additive* or *countably additive* (or neither).

Definition

A set function $\phi : \mathcal{R} \to [-\infty, \infty]$ is called an additive set function on $\mathcal{R}$ if and only if

$$\phi(A \cup B) = \phi(A) + \phi(B) \quad \text{whenever} \quad A \cap B = 0.$$

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Countably additive set functions

Definition

A set function $\phi : \mathcal{R} \to [-\infty, \infty]$ is called a countably additive set function on $\mathcal{R}$ if and only if

$$\phi(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} \phi(A_i), \tag{1}$$

whenever $A_i \cap A_j = 0, \ i \neq j$.

- The left hand side is ϕ of the union of sets, and ϕ is assumed to be well-defined on $\mathcal{R}$, so ϕ of the union must belong to the extended number system $[-\infty, \infty]$.
- So this says that the partial sums of the infinite series on the right hand side must either converge to something finite or $\sum_{i=1}^{n} \phi(A_i) \to \infty$ or $-\infty$ as $n \to \infty$ (e,g, the limit cannot oscillate, b/c of the previous bullet point).
- Hence we can write: $\sum_{i=1}^{n} \phi(A_i) \to \sum_{i=1}^{\infty} \phi(A_i)$ as $n \to \infty$ in both situations, i.e., if the series converges or if it diverges to $\pm\infty$.
- Because the left hand side of (1) is the same for any rearrangement of sets A_i, *if* the right hand side converges, it converges absolutely (Rudin p. 75).

Properties of the set functions ϕ

We note the following:

- We assume ϕ's range does not contain both ∞ and $-\infty$.
- We assume ϕ maps to a finite number at least for one set A.

If ϕ is additive, then

1. $\phi(0) = 0$. Proof: HW
2. $\phi(A_1 \cup \cdots A_n) = \phi(A_1) + \cdots + \phi(A_n)$ if $A_i \cap A_j = 0, \quad i \neq j$. Proof: obvious, by induction.
3. $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$ Proof: HW
4. If ϕ is nonnegative, i.e., $\phi(A) \geq 0$ for every A, and $A_1 \subset A_2$, then
$$\phi(A_1) \leq \phi(A_2)$$
Proof: HW
5. If $B \subset A$ and $|\phi(B)| < \infty$ then $\phi(A - B) = \phi(A) - \phi(B)$. Proof: HW.

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$$A = \cup_{n=1}^{\infty} A_n.$$

Then as $n \to \infty$, we have $\phi(A_n) \to \phi(A)$.

Proof: Let $B_1 = A_1, \quad B_n = A_n - A_{n-1}, \; n \geq 2.$
Then observe B_i's are pairwise disjoint and $A_n = B_1 \cup \cdots \cup B_n$.
Hence $\phi(A_n) = \phi(B_1 \cup \cdots \cup B_n)$, so by additivity of ϕ and B_n's being pairwise disjoint, we have
$\phi(A_n) = \sum_{i=1}^{n} \phi(B_i) \to \sum_{i=1}^{\infty} \phi(B_i) \quad$ as $n \to \infty$.
Now $A = \cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} (B_1 \cup \cdots \cup B_n) = \cup_{n=1}^{\infty} B_n$.
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Definition of a measure

Definition

Let $\mathcal{R}$ be a σ-ring. A (nonnegative) measure is a countably additive set function $\mu : \mathcal{R} \to [0, \infty]$.

- We note that one can also consider measures that are negative or complex. Also, measures can be defined on $\sigma-$*algebras* of sets instead of σ-rings (see for example big Rudin).
- Next week we will define a *measure space* and a *measurable space.*

Summary of definitions

We have defined the following: (fill in the definitions)

- ring of sets:
- σ-ring of sets:
- set function:
- additive set function:
- countably additive set function:
- measure:

Lebesgue measure construction

Henri Lebesgue

- Lebesgue measure constructed in 1901
- Lebesgue integral defined in 1902
- Both published in 1902 as part of Lebesgue's dissertation

Lebesgue measure construction: Step 0

The Lebesgue measure is defined in 6 steps, gradually increasing the complexity of sets considered. Note: each step is a definition.

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Intervals in $\mathbb{R}^n$

An interval in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ determined by two points $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n) \in R^n$. The points x belonging to the interval satisfy

$$a_i \le x_i \le b_i, \quad i = 1, \dots n. \tag{2}$$

- Intervals are also called n-cells in Rudin.
- If $n = 1$, the interval is
- If $n = 2$, the interval is
- if $n = 3$, the interval is
- Rudin also allows $\le$ to be replaced by $<$ in the definition of the interval. Jones does not, and calls the intervals *special rectangles*. We follow Rudin here as this makes things technically simpler in the future.

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Step 1: Lebesgue measure of intervals

1) Intervals: $m(I) = \Pi_{i=1}^{n}(b_i - a_i)$

- If $n = 1$, $m(I) =$ so the Lebesgue measure in this case is
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Step 2: Lebesgue measure of special polygons

2) Special polygons: a special polygon A is a finite union of non-overlapping (having disjoint interiors) *closed* intervals (where it is assumed that each interval has nonzero measure).
$m(A) = \sum_{i=1}^{k} m(I_i)$.
It can be shown $m(A)$ is independent of how we decompose A.

Step 3 and 4: Lebesgue measure of open and compact sets

3) Open sets, $G \subset \mathbb{R}^n$ open:
$m(G) = \sup\{m(E) : E \subset G, E \text{ a special polygon}\}$.
   - $m(\mathbb{R}^n) = \infty$. We can show $m(\mathbb{R}^n) \geq (2a)^n$ for any $a > 0$. (see the board)
   - m as defined on open sets is in general subadditive ($m(\cup_{i=1}^{\infty} G_i) \leq \Sigma_{i=1}^{\infty} m(G_i)$), and countably additive if the sets are pairwise disjoint (see Jones).

4) Compact sets: $K \subset \mathbb{R}^n$ compact:
$m(K) = \inf\{m(G) : K \subset G, G \text{ open}\}$.
   - What if K is a special polygon? To be consistent, we should check that the definition given in Step 2 agrees with the definition in Step 4. See Jones.

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Outer & Inner measures: Step 5

Before we go to the next step we define, *outer* and *inner* measures. Let A be an arbitrary subset in $\mathbb{R}^n$. Then

outer measure: $m^*(A) = \inf\{m(G) : A \subset G, G \text{ open}\}$

inner measure: $m_*(A) = \sup\{m(K) : K \subset A, K \text{ compact}\}$

Some of the properties:

- $m_*(A) \leq m^*(A)$
- $A \subset B$, then $m^*(A) \leq m^*(B)$ and $m_*(A) \leq m_*(B)$.
- If A is open or compact, then $m^*(A) = m_*(A) = m(A)$. (Compact: in class. Open: HW).

5) Arbitrary set $A \subset \mathbb{R}^n$ with a FINITE outer measure. We say A with a finite outer measure is Lebesgue measurable if and only if

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Equivalent characterization of Lebesgue measurable sets with finite outer measure

Theorem

Let $A \subset \mathbb{R}^n$ and $m^*(A) < \infty$. Then A is Lebesgue measurable if and only if for every $\epsilon > 0$, there exists a compact set K and an open set G such that

$$K \subset A \subset G, \quad \text{and} \quad m(G - K) < \epsilon.$$

Corollary

If $m_*(A) = m^*(A) < \infty$ and $m_*(B) = m^*(B) < \infty$, then the sets $A \cup B$, $A \cap B$ and $A - B$ are Lebesgue measurable and have a finite measure.

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Lebesgue measure of sets with finite outer measure is countably additive

Theorem

Let $A_i \subset \mathbb{R}^n$ and $m^*(A_i) < \infty$ and A_i is Lebesgue measurable. Suppose A is a set such that $m^*(A) < \infty$ and $A = \cup_{i=1}^{\infty} A_i$. Then A is Lebesgue measurable and

$$m(A) \leq \Sigma_{i=1}^{\infty} m(A_i).$$

If A_i's are pairwise disjoint then

$$m(A) = \Sigma_{i=1}^{\infty} m(A_i).$$

When is an arbitrary subset of $\mathbb{R}^n$ Lebesgue measurable?

Definition

6) An arbitrary set $A \subset \mathbb{R}^n$ is Lebesgue measurable if and only if $A \cap M$ is Lebesgue measurable for every measurable $M \subset \mathbb{R}^n$ where $m^*(M) < \infty$. The Lebesgue measure of A is then

$$m(A) = \sup\{m(A \cap M) : M \subset \mathbb{R}^n, m_*(M) = m^*(M) < \infty\}.$$

- Note the following: Since $A \cap M \subset M$ and $m^*(M) < \infty$, we have $m^*(A \cap M) < \infty$, so when we check if $A \cap M$ is Lebesgue measurable, we check it in the sense of the definition given in Step 5.

Consistency check

Theorem

Let $A \subset \mathbb{R}^n$ with $m^*(A) < \infty$. Then A is Lebesgue measurable according to definition in Step 5 if and only if it is Lebesgue measurable according to the definition in Step 6. Moreover, $m(A)$ in Step 5 produces the same number as $m(A)$ in Step 6.

Proof of the Consistency check Thm

Proof.

Suppose $m^*(A) < \infty$, and A is measurable according to the definition in Step 5. Then if M is another set that is measurable with $m^*(M) < \infty$ we have $A \cap M$ is measurable by the Corollary.

Next suppose A is measurable according to the definition in Step 6. Consider $B_k(0)$, an open ball of radius k centered at the origin. $m(B_k) < \infty$. So since A is measurable according to the definition in Step 6, $A \cap B_k$ is measurable and $m(A \cap B_k) < \infty$ since $A \cap B_k \subset B_k$. Now we can write A as $A = \cup_{k=1}^{\infty}(A \cap B_k)$, so by the countably additive property of the measure defined in Step 5, we have A is measurable. Now we show that the two definitions produce same value for $m(A)$. Let $\bar{m}(A)$ denote the measure defined in Step 6:

$$\bar{m}(A) = \sup\{m(A \cap M) : M \subset \mathbb{R}^n, m_*(M) = m^*(M) < \infty.\}$$

Since $A \cap M \subset A$ we have $m(A \cap M) \le m(A)$ so by definition of $\sup$, $\bar{m}(A) \le m(A)$. But since $m(A) < \infty$, we can let $M = A$, so again by definition of $\sup$, $\bar{m}(A) \ge m(A \cap A) = m(A)$, so $\bar{m}(A) = m(A)$, as needed. □

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Properties of Lebesgue measure

Let $\mathcal{L}$ denote the set of all Lebesgue measurable subsets of $\mathbb{R}^n$.

1. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$.
2. Countable unions and countable intersections of measurable sets are measurable.
3. If $A, B \in \mathcal{L}$, then $A - B \in \mathcal{L}$.
4. If $A_k \in \mathcal{L}$, then $m(\cup_{k=1}^{\infty} A_k) \leq \Sigma_{k=1}^{\infty} m(A_k)$ and if A_k are pairwise disjoint, then
$$m(\cup_{k=1}^{\infty} A_k) = \Sigma_{k=1}^{\infty} m(A_k).$$
5. If $A_1 \subset A_2 \subset \dots$, and A_k are measurable, then $m(\cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$ (we showed this already for countably additive set functions (see Thm 11.3 in Rudin or these notes), and m is countably additive by the previous property)
6. If $A_1 \supset A_2 \supset \dots$, A_k are measurable, and $m(A_1) < \infty$, then $m(\cap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} m(A_k)$.
7. All open sets and all closed sets are measurable.
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Properties of Lebesgue measure

Let $\mathcal{L}$ denote the set of all Lebesgue measurable subsets of $\mathbb{R}^n$.

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Proof of: if $m^*(A) = 0$, then A is measurable and $m(A) = 0$.

By properties of inner and outer measure we have

$$0 \leq m_*(A) \leq m^*(A).$$

But since $m^*(A) = 0$, we must have $m_*(A) = 0$. So

$$m_*(A) = m^*(A) = 0,$$

so A is measurable (using definition from Step 5 since the outer measure is finite.)

More properties of the Lebesgue measure

9) If A is measurable, then $m^*(A) = m_*(A) = m(A)$.
10) $A \subset \mathbb{R}^n$ is Lebesgue measurable if and only if for every $\epsilon > 0$, there exists a closed set K and an open set G such that

$$K \subset A \subset G, \quad \text{and} \quad m(G - K) < \epsilon.$$

another equivalent definition of Lebesgue measurable due to Carathéodory

Theorem

A is measurable if and only if for every set $E \subset \mathbb{R}^n$

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c).$$