Math 2001
Truth tables and logical equivalence

1. Write a truth table for the logical statement $\sim (P \vee Q) \implies (\sim P \wedge Q)$. Do it step by step (i.e. include columns for $\sim P$ and $P \vee Q$ etc., building up to the full expression).

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$\sim P \wedge Q$	$\sim(P \vee Q) \Rightarrow \sim P \wedge Q$
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	F	F

2. Is the example in the previous problem true? (Hint: Is $P \wedge Q$ true? Is $P \vee Q$ true?)

The above statement is not always true (not a tautology), since when P and Q both take on the truth value F, the entire statement is false.

3. Two statments are *logically equivalent* if they have the same truth values regardless of the input values of the variables. Name a much simpler expression that is logically equivalent to the expression in the first example.

$P \vee Q$

4. Prove DeMorgan's Laws:

- $\sim (P \wedge Q) = (\sim P) \vee (\sim Q)$,
- $\sim (P \vee Q) = (\sim P) \wedge (\sim Q)$.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P \vee \sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$\sim P \wedge \sim Q$
T	T	F	F	T	F	F	T	F	F
T	F	F	T	F	T	T	T	F	F
F	T	T	F	F	T	T	T	F	F
F	F	T	T	F	T	T	F	T	T

We can see that the truth values of $\sim P \vee \sim Q$ and $\sim(P \wedge Q)$ are the same given any truth assignment for P and Q. Thus $\sim(P \wedge Q)$ is logically equivalent to $\sim P \vee \sim Q$.

Similarly, since the truth values for $\sim(P \vee Q)$ and $\sim P \wedge \sim Q$ are the same for any truth assignment for the variables P and Q, $\sim(P \vee Q)$ is logically equivalent to $\sim P \wedge \sim Q$.

5. In Section 2.4 you saw that $P \Leftrightarrow Q$ is logically equivalent to $(P \implies Q) \wedge (Q \implies P)$. Thus, in some sense $\Leftrightarrow$ isn't needed – it can be 'generated' by $\implies$ and $\wedge$. Now find a way to generate $P \implies Q$ using only $\vee$, $\wedge$ and $\sim$. (This shows that in some sense $\Leftrightarrow$ isn't needed either!).

The following truth table shows the logical equivalence of $P \Rightarrow Q$ and $\sim P \vee Q$:

P	Q	$\sim P$	$\sim P \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

So we have 'generated' $\Rightarrow$ by $\sim$ and $\vee$.

6. A *tautology* is a Boolean expression that evaluates to TRUE for all possible values of its variables. Work together (as always) to come up with an example of a tautology in two variables (you might try one variable first if you are stuck). Provide a proof (that is, a truth table) that it is a tautology.

The following truth table shows that $(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow Q)$ is a tautology.

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	T	T	T

All true!

7. A *contradiction* is a Boolean expression that evaluates to FALSE for all possible values of its variables. Come up with an example of a contradiction in two variables and prove that it is one.

$(P \wedge \sim Q) \Leftrightarrow (P \Rightarrow Q)$ is a contradiction.

P	Q	$\sim Q$	$P \Rightarrow Q$	$P \wedge \sim Q$	$(P \wedge \sim Q) \Leftrightarrow (P \Rightarrow Q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

All false!

8. How many lines (besides the header) does the truth table for a Boolean expression in 8 variables have?

Since there are 2 possible values for each variable, there are $\underbrace{2 \times 2 \times \cdots \times 2}_{8 \text{ times}} = 2^8$ lines in the truth table.

9. How many logically distinct Boolean expression could you define on two variables? Writing out all possibilities is possible but a hassle. Instead figure out how to count the possibilities.

Realizing that two expressions are logically equivalent iff any truth assignment to the variables gives the same truth value for both expressions, we see that for each distinct truth assignment for the variables there are only 2 options – either a statement is true or it is false. Since there are $2^2 = 4$ truth assignments on 2 variables, we have $2^{(4)} = 16$ logically distinct boolean expressions on 2 variables.

10. Can all of the boolean expressions on two variables be constructed using only $\wedge$, $\vee$, and $\sim$? In other words, are all boolean expressions on two variables logically equivalent to one that combines only $\wedge$, $\vee$, and $\sim$? Why or why not? Yes. We have already seen that $\wedge$ and $\Rightarrow$ 'generate' $\Leftrightarrow$, and that $\sim$ and $\vee$ 'generate' $\Rightarrow$, so we can write an equivalent statement to one involving $\wedge, \vee, \sim, \Rightarrow$, and $\Leftrightarrow$ just using $\wedge, \vee$, and $\sim$. But this means we can write a statement that is equivalent to any Boolean expression using only $\wedge, \vee$, and $\sim$. (For any number of variables!)

11. How many logically distinct Boolean expressions could you define on n variables?

By the same argument as problem 9, there are $2^{(2^n)}$ logically distinct Boolean expressions on n variables.