SOLUTIONS

Negating Statements

1. We can use DeMorgan's Laws to negate statements that contain "$\wedge$" or "$\vee$".

   (a) Write down DeMorgan's Laws.

   $\sim(P \wedge Q) = \sim P \vee \sim Q$.

   $\sim(P \vee Q) = \sim P \wedge \sim Q$.

   (b) Negate the open sentence $x > 0 \wedge y \leq 0$.

   $\sim(x > 0 \wedge y \leq 0) = \sim(x > 0) \vee \sim(y \leq 0) = x \leq 0 \vee y > 0$

   (c) Negate the statement "The Broncos win this week or I'll eat my hat".

   $\sim$(The Broncos win this week or I'll eat my hat)

   $=$ $\sim$(The Broncos win this week) $\wedge$ $\sim$(I'll eat my hat)

2. We can negate statements that have quantifiers.

   (a) Negate the statement $\forall x \in S, P(x)$

   $\exists x \in S, \sim P(x)$

   (b) Negate the statement $\exists x \in S, Q(x)$

   $\forall x \in S, \sim Q(x)$

   (c) Negate the statement "All primes are odd".

   Let P be the set of all primes. The statement is $\forall x \in P$, x is odd.

   The Negation is $\exists x \in P$, x is even.

   (d) Negate the statement "There is a linear function $f(x)$ that does not go through the origin".

   Let F = set of all linear functions. statement: $\exists f \in F, f(0) \neq 0$.

   Negation: $\forall f \in F, f(0) = 0$.

3. We can negate statements that have multiple quantifiers.

   (a) Negate the statement $\forall x \in S, \exists y \in T, P(x, y)$.

   $\sim(\forall x \in S, \exists y \in T, P(x,y)) = \exists x \in S, \sim[\exists y \in T, P(x,y)]$

   $= \exists x \in S, \forall y \in T, \sim P(x,y)$

   (b) Negate the statement $\exists x \in S, \forall y \in T, P(x, y)$.

   $\sim(\exists x \in S, \forall y \in T, P(x,y)) = \forall x \in S, \sim(\forall y \in T, P(x,y))$

   $= \forall x \in S, \exists y \in T, \sim P(x,y)$

   (c) Negate the statement $\forall n \in \mathbb{Z}, \exists X \subset \mathbb{N}, |X| = n$.

   $\exists n \in \mathbb{Z}, \forall X \subset \mathbb{N}, |X| \neq n$.

   (d) Negate the statement $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n + 5$.

   $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m \neq n + 5$

   (e) Negate the statement "For every real number x, there is a real number y for which $y^2 = x$".

   Statement: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^2 = x$.

   Negation: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 \neq x$.

4. We can negate conditional and biconditional statments.

   (a) Review: Express $P \Rightarrow Q$ using only nots, ands or ors.

   $\sim P \vee Q$

   (b) Use what you wrote down above to negate $P \Rightarrow Q$.

   $\sim (P \Rightarrow Q) = \sim(\sim P \vee Q) = P \wedge \sim Q$

   (c) Now write $P \Leftrightarrow Q$ using only nots, ands or ors (hint: $P \Leftrightarrow Q = (P \Rightarrow Q) \wedge (Q \Rightarrow P)$).

   $(P \Leftrightarrow Q) = ((P \Rightarrow Q) \wedge (Q \Rightarrow P)) = (\sim P \vee Q) \wedge (\sim Q \vee P).$

   (d) Negate $P \Leftrightarrow Q$.

   $\sim((\sim P \vee Q) \wedge (\sim Q \vee P)) = \sim(\sim P \vee Q) \vee \sim(\sim Q \vee P))$
   $= (P \wedge \sim Q) \vee (Q \wedge \sim P)$

5. Remember from Section 2.8 that conditional statements sometimes have an implicit universal quantifier in them. You have to remember that when you are negating conditional statements.

   (a) For example, "If p is odd, then p^2 is odd" can be translated to $\forall p \in \mathbb{Z}$, if p is odd then p^2 is odd. Negate this statement.

   $\exists p \in \mathbb{Z}, \sim(p \text{ odd} \Rightarrow p^2 \text{ odd}) = \exists p \in \mathbb{Z}, \sim(p \text{ not odd or } p^2 \text{ odd})$
   $= \exists p \in \mathbb{Z}, p \text{ odd and } p^2 \text{ not odd}$

   (b) Negate the statement "If $x \in R$, then $x^2 > 0$"

   statement: $\forall x \in \mathbb{R}, x^2 > 0$
   Negation: $\exists x \in \mathbb{R}, x^2 \leq 0.$

   (c) Not all conditional statements have an implicit universal quantifier. Negate the statement "If all zebras have stripes, then all okapis have no stripes"

   Negation: all zebras have stripes and not (all okapis have no stripe
   = all zebras have stripes and there exists a striped okapi.

6. When we put it all together, it can get gnarly. Sometimes it helps to translate the sentence into symbolic logic before you begin negating.

   (a) Negate the statement "If $n \in \mathbb{N}$, then $12|(n^4 - n^2)$".

   statement: $\forall n \in \mathbb{N}, \exists c \in \mathbb{Z}, 12c = n^4 - n^2$
   negation: $\exists n \in \mathbb{N}, \forall c \in \mathbb{Z}, 12c \neq n^4 - n^2$

   (b) Assume for this problem that $f(x)$ is a fixed function. Negate the statement "For all $N \in \mathbb{N}$ there is a real positive number δ such that if $|x - 5| < \delta$ then $|f(x)| > N$".

   statement: $\forall N \in \mathbb{N}, \exists \delta > 0$, if $|x-5| < \delta$ then $|f(x)| > N$
   $= \forall N \in \mathbb{N}, \exists \delta > 0, \forall x \in \mathbb{R}$, if $|x-5| < \delta$ then $|f(x)| > N.$
   Negation: $\exists N \in \mathbb{N}, \forall \delta > 0, \exists x \in \mathbb{R}, |x-5| < \delta$ and $|f(x)| \leq N$