Functions, Part 3

- Definition: Suppose $f : A \mapsto B$ and $g : B \mapsto C$ are functions. The *composition* of f with g is a function $g \circ f : A \mapsto C$. For $x \in A$, $g \circ f(x) = g(f(x))$.
- Definition: Given a set A, the *identity function* on A is the function $i_A : A \mapsto A$ defined as $i_A(x) = x$ for every $x \in A$.
- Given a relation R from A to B, the *inverse relation of* R is the relation from B to A defined as $R^{-1} = \{(y, x) : (x, y) \in R\}$. In other words, the inverse of R is the relation R^{-1} obtained by interchanging the elements in every ordered pair in R.
- Theorem: The function $f : A \mapsto B$ is bijective if and only if the inverse relation f^{-1} is a function from B to A.
- Definition: If $f : A \mapsto B$ is bijective, then its *inverse* is the function $f^{-1} : B \mapsto A$.
- Key result regarding f and f^{-1}: The functons f and f^{-1} satisfy the equations $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Example 1: Visual representation of composition

$g \circ f(x) = g(f(x))$ $\qquad g(f(x)) \leftarrow \boxed{g} \xleftarrow{f(x)} \boxed{f} \leftarrow x$

Example 2: Composition using formulas

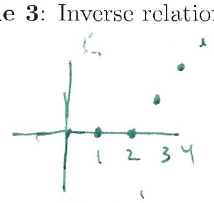
$f(x) = x^2 \qquad g(x) = 2x - 1$

$f \circ g(x) = (2x-1)^2$

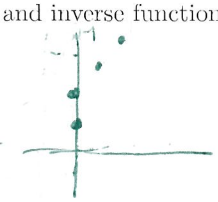
$g \circ f(x) = 2x^2 - 1$

$f \circ g \circ f(x) = (2x^2-1)^2$

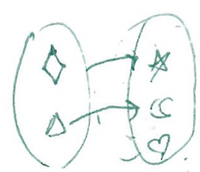
Example 3: Inverse relations and inverse functions



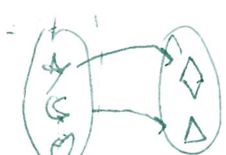
relation from N to N



inverse relation. not a function.



Relation from (◇, △) to (✡, ☾, ♡)

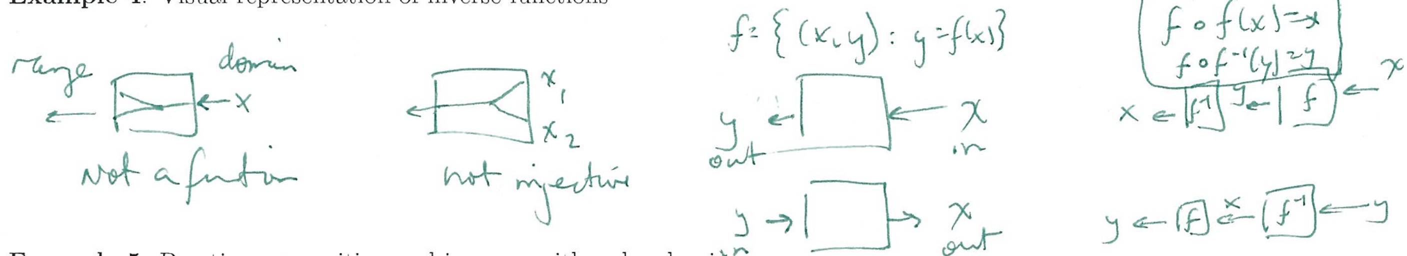


inverse relation not a function on (✡, ☾, ♡)

a relation is ...

injective & surjective ⟺ its inverse is a function.

Example 4: Visual representation of inverse functions



Example 5: Practice composition and inverses with ordered pairs

Suppose that $A = \{1, 2, 3\}$, $f : A \mapsto A$ and $g : A \mapsto A$, $f = \{(1, 3), (2, 1), (3, 2)\}$ and $g = \{(1, 2), (2, 3), (3, 2)\}$.

$f \circ g =$ $\{(1,1), (2,2), (3,1)\}$

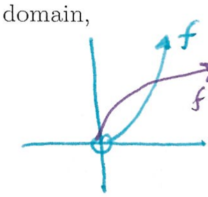
$f \circ f \circ f =$ $\{(1,1), (2,2), (3,3)\}$

If f is bijective, then find f^{-1} $f^{-1} = \{(1,2), (2,3), (3,1)\}$

If g is bijective, then find g^{-1} g is not bijective, since $g(1) = g(3) = 2$.

Example 6: Practice finding inverses with formulas and graphs

Suppose that both the domain and codomain of f are the positive reals, and say that for all x in the domain, $f(x) = x^2$. Is f invertible? If so, then find a formula for f^{-1}. Graph f and f^{-1} on the same graph.



f is "onto" the positive reals;

if $y \in \mathbb{R}^+$, Let $x = \sqrt{y}$. $f(x) = (\sqrt{y})^2 = y$.

So f is surjective.

f is injective (one-to-one):

if $f(x_1) = f(x_2)$, then $x_1^2 = x_2^2 \Rightarrow |x_1| = |x_2|$.

But $x_1 \geq 0$ & $x_2 \geq 0$, so $x_1 = x_2$. f is injective

f is bijective & thus invertible.

$f^{-1}(x) = \sqrt{x}$.

Example 7: Practice finding inverses with formulas

Suppose that $f : \mathbb{R} - \{2\} \mapsto \mathbb{R} - \{1\}$ by $f(x) = \frac{x+1}{x-2}$. Find a formula for $f^{-1}(x)$. Then confirm algebraically that $f^{-1} \circ f = i_A$.

we previously showed f is bijective (see Functions part 2)

Now if $y \in \mathbb{R} - \{1\}$, If $f(x) = y$ then

$$\frac{x+1}{x-2} = y \Rightarrow (x-2)y = (x+1) \Rightarrow xy - 2y = x + 1$$

$$\Rightarrow xy - x = 2y + 1 \Rightarrow x(y-1) = 2y + 1$$

$$\Rightarrow x = \frac{2y+1}{y-1} \Rightarrow f^{-1}(y) = \frac{2y+1}{y-1}.$$

Say $x \in \mathbb{R} - \{2\}$. $f^{-1} \circ f(x) = f^{-1}\left(\frac{x+1}{x-2}\right) = \dfrac{2\left(\frac{x+1}{x-2}\right) + 1}{\frac{x+1}{x-2} - 1}$

$$= \frac{2x + 2 + x - 2}{x + 1 - x + 2} = \frac{3x}{3} = x. \quad \text{So } f^{-1} \circ f(x) = x, \ \forall x \in A$$

Thus $f^{-1} \circ f = id_A$