Functions, Part 2

- Definition: A *function* f from a set A into a set B (written $f : A \mapsto B$) is a relation $f \subseteq A \times B$ satisfying the property that for all $a \in A$ the relation f contains exactly one ordered pair of the form (a, b). The statement $(a, b) \in f$ is also written $f(a) = b$. The set A is called the domain, and B is called the codomain.
- A function $f : A \mapsto B$ is *surjective* (or "onto") if for every $b \in B$ there is an $a \in A$ with $f(a) = b$.
- A function $f : A \mapsto B$ is *injective* (or "one-to-one") if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$.
- A function $f : A \mapsto B$ is *bijective* (or "one-to-one and onto") if it is both surjective and bijective.
- Here are two ways to show that a function $f : A \mapsto B$ is injective:
  1. Direct approach: Suppose $x, y \in A$ and $x \neq y$. Show that $f(x) \neq f(y)$.
  2. Contrapositive approach: Suppose $x, y \in A$ and $f(x) = f(y)$. Show that $x = y$.
- To show that a function $f : A \mapsto B$ is surjective: suppose that $b \in B$. Prove that there is an $a \in A$ for which $f(a) = b$.
- Suppose $f : A \mapsto B$ is a function.
  1. If $X \subseteq A$, the *image* of X is the set $f(X) = \{f(x) : x \in X\} \subseteq B$.
  2. If $Y \subseteq B$, the *preimage* of Y is the set $f^{-1}(Y) = \{x \in A : f(x) \in Y\} \subseteq A$.
  3. $f(A)$ is called the *range* of f.
- TO show f is not injective: find an $x, y \in A$, $x \neq y$, $f(x) = f(y)$
- TO show f is not surjective: Find $b \in B$, $\forall a \in A$, $f(a) \neq b$.

1. Consider the function $f : \mathbb{N} \mapsto \mathbb{N}$ defined by the formula $f(n) = n+3$. Determine the range of f. Find $f(\{1, 2, 3\})$. Find $f^{-1}(\{1, 2, 3\})$. Determine if f is surjective, injective, bijective, providing a full proof for each of these.

range of $f = f(\mathbb{N}) = \{4, 5, 6, 7, \ldots\}$

$f(\{1, 2, 3\}) = \{4, 5, 6\}$.

$f^{-1}(\{1, 2, 3\}) = \emptyset$

f is injective: say $f(x) = f(y)$. then $x+3 = y+3 \Rightarrow x = y$.
so f is injective.

f is not surjective. consider $y = 1$. if $f(x) = y$, then
$x + 3 = 1$
$x = -2$. But $x \in \mathbb{N}$, a contradiction.
so $y = 1 \notin$ range of f.
so f is not surjective.

f is not surjective, so it is not bijective.

2. Consider the function $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $f(n) = |n|$. Determine the range of f. Find $f(\mathbb{N})$. Find $f^{-1}(\mathbb{N})$. Determine if f is surjective, injective, bijective, providing a full proof for each of these.

range of $f = f(\mathbb{Z}) = \{0, 1, 2, \ldots\} = \{x \in \mathbb{Z} : x \geq 0\}$

$f(\mathbb{N}) = \{1, 2, 3, \ldots\} = \mathbb{N}$

$f^{-1}(\mathbb{N}) = \{\ldots -3, -2, -1, 1, 2, 3, \ldots\} = \mathbb{Z} - \{0\}$

f is not injective, because if $x = 1$ and $y = -1$, then $f(x) = f(y) = 1$, but $x \neq y$. So f is not injective.

f is not surjective: Consider $-1 \in \mathbb{Z}$, the codomain. But $\forall n \in \mathbb{Z}$, $f(n) = |n| \geq 0$. So $f(n) \neq -1$. -1 is not in the range of f. So f is not surjective.

f is neither surjective nor injective, so it is not bijective.

3. Consider the function $f : \mathbb{R} - \{2\} \mapsto \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x+1}{x-2}$. Give an example of an element in the set $f^{-1}(\mathbb{N})$. Determine if f is surjective, injective, bijective, providing a full proof for each of these.

$f(3) = \frac{4}{1} = 4 \in \mathbb{N}$. So $3 \in f^{-1}(\mathbb{N})$.

f is injective. Proof: Say $f(x) = f(y)$, $x, y \in \mathbb{R} - \{2\}$.

$\frac{x+1}{x-2} = \frac{y+1}{y-2} \Rightarrow (x+1)(y-2) = (y+1)(x-2)$

$\Rightarrow xy + y - 2x - 2 = xy + x - 2y - 2$

$\Rightarrow 3y = 3x \Rightarrow x = y$. So f is injective.

f is surjective. Proof: Say $y \in \mathbb{R} - \{1\}$. Let $x = \frac{2y+1}{y-1}$

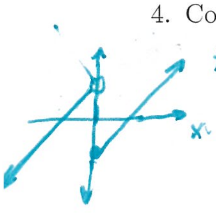
If $x = 2$, then $\frac{2y+1}{y-1} = 2 \Rightarrow 2y - 2 = 2y + 1 \Rightarrow -2 = 1$.

we conclude $x \neq 2$, so $x \in \mathbb{R} - \{2\}$.

$f(x) = \dfrac{\frac{2y+1}{y-1} + 1}{\frac{2y+1}{y-1} - 2} = \dfrac{2y+1+y-1}{2y+1-2(y-1)}$ (by clearing denominators)

$= \frac{3y}{3} = y$. So f is surjective.

4. Come up with an example of a function $f : \mathbb{R} \mapsto \mathbb{R}$ that is surjective but not injective.



$f(x) = \begin{cases} x - 1 & x \geq 0 \\ x + 1 & x < 0 \end{cases}$

Note that $f(1) = 0$ & $f(-1) = 0$, so f is not injective.

now if $y \geq 0$, Let $x = y + 1$. $x \geq 1$, so $f(x) = x - 1 = y + 1 - 1 = y$

if $y < 0$, Let $x = y - 1$. $x < 1$, so $f(x) = x + 1 = y - 1 + 1 = y$.

surjective.