EXAMPLE OF PROOF BY STRONG INDUCTION:

CONSIDER THE FIBONACCI SEQUENCE:

$1, 1, 2, 3, 5, \ldots$

(EACH TERM IS FOUND RECURSIVELY BY ADDING THE PREVIOUS TWO TERMS).

THERE IS AN EXPLICIT FORMULA FOR THIS SEQUENCE. THE FOLLOWING PROPOSITION GIVES THAT FORMULA.

PROPOSITION: IF $F_1 = F_2 = 1$, AND $F_n = F_{n-1} + F_{n-2}$, THEN

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

PROOF: (by strong induction)

STEP 1: SHOW THE FORMULA IS CORRECT FOR $n=1$ and $n=2$:

$$n=1: \quad \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = \frac{1}{\sqrt{5}}(\sqrt{5}) = 1 = F_1$$

$$n=2: \quad \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{1}{4}\left(1+2\sqrt{5}+5 - (1-2\sqrt{5}+5)\right)\right]$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{4} \cdot 4\sqrt{5} = 1 = F_2.$$

THE STATEMENT IS TRUE FOR $n=1$ AND $n=2$

STEP 2: SUPPOSE $K \geq 2$, AND THE STATEMENT IS TRUE FOR $n = 1, 2, \ldots, K$.

WE MUST SHOW THE STATEMENT IS TRUE FOR $n = K+1$.

$$F_{K+1} = F_K + F_{K-1}$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^K}{2^K} - \frac{(1-\sqrt{5})^K}{2^K}\right] + \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{K-1}}{2^{K-1}} - \frac{(1-\sqrt{5})^{K-1}}{2^{K-1}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{K-1}(1+\sqrt{5}) + 2(1+\sqrt{5})^{K-1}}{2^K} - \frac{(1-\sqrt{5})^{K-1}\cdot(1-\sqrt{5}) + 2(1-\sqrt{5})^{K-1}}{2^K}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{K-1}(3+\sqrt{5})}{2^K} - \frac{(1-\sqrt{5})^{K-1}(3-\sqrt{5})}{2^K}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{K-1}(1+\sqrt{5})^2}{2^{K+1}} - \frac{(1-\sqrt{5})^{K-1}(1-\sqrt{5})^2}{2^{K+1}}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})^{K+1}}{2^{K+1}} - \frac{(1-\sqrt{5})^{K+1}}{2^{K+1}}\right]$$

NOTE:

$(1+\sqrt{5})^2 = 1+2\sqrt{5}+5$

$= 6+2\sqrt{5} = 2(3+\sqrt{5})$

$(1-\sqrt{5})^2 = 1-2\sqrt{5}+5$

$= -2\sqrt{5}+6$

$= 2(3-\sqrt{5})$

THE FORMULA IS CORRECT FOR $n = K+1$

STEP 3: BY STRONG INDUCTION, THE FORMULA IS CORRECT FOR ALL $n \in \mathbb{N}$.