Equivalence Relations and Equivalence Classes

Recall from the last worksheet that an equivalence relation is a relation that is reflexive, symmetric and transitive.

1. Which of the following relations are reflexive, which are symmetric, and which are transitive? Which are equivalence relations?

   (a) The relation "has the same cardinality" on the power set of $\mathbb{N}$

   Equivalence relation (reflexive, symmetric & transitive)

   (b) The relation $<$ on the real numbers

   reflexive : no
   symmetric : no
   transitive : yes

   (c) The relation "is genetically related to" on the set of humans

   reflexive : yes
   symmetric : yes
   transitive : yes

   (d) The relation $\subset$ on the power set of $\mathbb{Z}$

   reflexive : yes
   Symmetric : no
   transitive : yes

2. Consider the following definition: Definition: Suppose R is an equivalnce relation on a set A. Given any element $a \in A$, the *equivalence class containing* a, written $[a]$, is the set $\{x \in A : xRa\}$. In other words, it is the set of all elements of A that relate to a.

   (a) List the equivalence classes of the relation $=$ on the set of natural numbers.

   Each element of $\mathbb{N}$ is in its own equivalence class

   $\{1\}, \{2\}, \{3\}, \ldots$

   (b) List the equivalence classes of the relation "congruent mod 7".

   $[0] = \{\ldots -7, 0, 7, 14, \ldots\}$; $[1] = \{\ldots, -6, 1, 8, \ldots\}$; $[2] = \{\ldots -5, 2, 9, \ldots\}$
   $[3] = \{\ldots -4, 3, 10, \ldots\}$; $[4] = \{\ldots -3, 4, 11, \ldots\}$; $[5] = \{\ldots, -2, 5, 12, \ldots\}$
   $[6] = \{\ldots, -1, 6, 13, \ldots\}$

   (c) List the equivalence classes of the relation

   $$R = \{(a, b) : a, b \in \mathbb{Z}, a - b \text{ is even}\}$$

   $[0] = \{n : n \text{ is even}\}$
   $[1] = \{n : n \text{ is odd}\}$

3. Prove the following theorem:
Theorem: Suppose R is an equivalence relation on a set A. Suppose that $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

See proof of Theorem 11.1 in textbook

4. This problem concerns the idea of a partition. Recall the definition:
Definition: A *partition* of a set A is a set of non-empty subsets of A, such that the union of all the subsets equals A, and the intersection of any two different subsets is $\emptyset$.

The following theorem is proved in the textbook: If R is an equivalence relation on a set A, then the set of equivalence classes of R forms a partition of A.

The theorem states that an equivalence relation on A automatically defines a partition. It is also true that any partition of A defines an equivalence relation on A. How would I use the partition to determine if xRy?

yes. This is discussed on page 190, & exercise 4 of the text.

define xRy if x and y are in the same subset of the partition.