Counting Worksheet 1 (order matters)

1. Consider lists with symbols from the list $\{S, B, U, C, \heartsuit, I\}$

   (a) How many lists of length 6 are there, assuming repetition is allowed?

   $$\underbrace{\underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6} \times \underline{6}}_{} = 6^6$$

   6 choices for each item in list (order matters, there is replacement)

   (b) How many lists of length 6 are there, assuming repetition is not allowed?

   $$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 6!$$

   no replacement, so 6 choices for 1st item, then only 5 remain for 2nd item, etc.

   (c) How many lists of length 4 are there, assuming repetition is allowed?

   order matters, replacement, 6 choices for each item.

   $$\underline{6} \times \underline{6} \times \underline{6} \times \underline{6} = 6^4$$

   (d) How many lists of length 4 are there, assuming repetition is not allowed?

   order matters, no replacement
   number of choices goes down by 1 each item.

   $$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} = \frac{6!}{2!}$$

   (e) How many lists of length 7 are there, assuming repetition is not allowed?

   $$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} \times \underline{0} = 0$$

   None! I run out of symbols!

2. In this year's Preakness Stakes horserace, 11 horses ran. How many ways could we have seen 1st, 2nd and 3rd place finishers?

   order matters, no replacement.

   $$\underline{11} \times \underline{10} \times \underline{9} = \frac{11!}{8!}$$

3. Calculate $\frac{101!}{98!}$ (no calculators allowed).

   $$\frac{101!}{98!} = \frac{101 \cdot 100 \cdot 99 \cdot 98!}{98!} = 101 \times 100 \times 99$$

   $$= 101 \times 9900 = 999900$$

4. How many ways are there to write a list of length r, choosing from a set of n elements, not allowing repetition?

$$\underline{n} \times \underline{n-1} \times \underline{n-2} \times \underline{n-3} \times \ldots \times \underline{n-r+1} = \frac{n!}{(n-r)!}$$

This is the general formula for ${}_nP_r = \frac{n!}{(n-r)!}$

5. How many ways are there to write a list of length r, choosing from a set of n elements, allowing repetition?

n choices for each of the r items

$$\underbrace{\underline{n} \cdot \underline{n} \cdot \underline{n} \cdot \underline{n} \ldots \underline{n}}_{r \text{ of them}} = n^r$$

6. In this problem we will make lists of letters from the English alphabet (26 letters, 5 of which are vowels).

(a) How many ways are there to list the entire alphabet?

order matters, no repetition, $26!$

(b) How many ways are there to list the entire alphabet if the vowels must all come at the end?

order matters, no repetition

21 consonants in order: $21!$ choices

5 vowels in order: $5!$ choices.

Total number of lists: $21!\,5!$

7. A combination lock has 4 wheels, containing the letters $\{a, b, c, d, e, f\}$.

(a) How many codes are there?

order matters, repetition allowed.

$$\underline{6} \times \underline{6} \times \underline{6} \times \underline{6} = 6^4$$

(b) How many codes are there that have no duplicate?

order matters, no repetition

$$\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} = \frac{6!}{2!}$$

(c) How many codes are there that have no two consecutive letters repeated?

order matters, repetition allowed with restriction.

$$\frac{6}{\text{any of the 6}} \times \frac{5}{\text{different from previous}} \times \frac{5}{\text{different from previous}} \times \frac{5}{\text{different from previous}} = 6 \cdot 5^3$$

8. Take a standard card deck (number cards 2, 3, 4, 5, 6, 7, 8, 9, 10, face cards J, Q, K, A, each of those 13 options showing up in each of the four suits ♡, ♠, ♣, ♢, for a total of 52 cards). Shuffle the deck and deal 5 cards, laying them down in order. Call this a "hand"

   (a) How many different hands are there?

   order matters, no repetition

   $$\underline{52} \times \underline{51} \times \underline{50} \times \underline{49} \times 48 = \frac{52!}{47!}$$

   (b) How many of those hands have all the same suit?

   order matters, no repetition

   For each suit: $\underline{13} \times \underline{12} \times \underline{11} \times \underline{10} \times \underline{9} = \frac{13!}{8!}$

   There are 4 possible suits, total number of "hands" is $4 \times \frac{13!}{9!}$

   (c) How many contain 4-of-a-kind?

   Note that 5-of-a-kind is not possible (no replacement, & there are only 4 of each kind). So for a 4-of-a-kind, one card is different. There are 13 choices for the number value. There are 48 choices for the "different card" and it can go in 5 possible spots. There are 4·3·2·1 ways to arrange the 4-of-a-kind in the remaining 4 slots. Total $= 13 \cdot 48 \cdot 5 \cdot 4!$

9. Take a standard deck, but this time draw a random card and write down what you draw. Return the card, shuffle again, and draw again, etc., until you have a list of 5 cards.

   (a) How many possible outcomes are there?

   order matters, but this time repetition is allowed.

   $$\underline{52} \times \underline{52} \times \underline{52} \times \underline{52} \times \underline{52} = 52^5$$

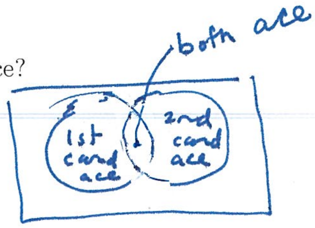
   (b) How many outcomes are there for which the 1st and 2nd card is an Ace?

   4 choices for each of the 1st 2 slots, 52 choices for the rest.

   $$\underline{4} \times \underline{4} \times \underline{52} \times \underline{52} \times \underline{52} = 4^2 \cdot 52^3$$

use calculator to show these are numerically the same

(c) How many outcomes are there for which the 1st card is an Ace or the 2nd card is an Ace?



method 1 replacement allowed, order matters.

1st card ace: $4\cdot4\cdot52\cdot52\cdot52$

2nd card ace: $52\cdot4\cdot52\cdot52\cdot52$

both of these count when both are aces.

both aces $= 4\cdot4\cdot52\cdot52\cdot52$

I double-counted the 2-ace situation. Total: $2\cdot4^2\cdot52^3-4^2\cdot52^3$

method 2: all possibilities, minus # of ways of getting no aces in 1st 2 slots. This gives $52^5-48\cdot48\cdot52\cdot52\cdot52$.

(d) How many outcomes are there that include a 4-of-a-kind?

I could have 5 of a kind, 13 choices for number values, 4^5 possible ways for each number value. $13\cdot4^5$ ways to get 5-of a kind.

Exactly 4-of-a-kind:
48 choices for different card, 5 places to put it. 13 choices for number value for 4-of-a-kind, 4^4 choices for the 4-of a kind.

Total: $13\cdot4^5+48\cdot5\cdot13\times4^4$

10. Consider 7-digit phone numbers.

(a) How many possibilities are there?

$\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}\times\underline{10}=10^7$

(b) How many possibilities are there that have at least one repeated digit?

count all possibilities, subtract ones with no repeated digit

no repeated digit: $\underline{10}\times\underline{9}\times\underline{8}\times\underline{7}\times\underline{6}\times\underline{5}\times\underline{4}=\frac{10!}{3!}$

at least one repeated digit: $10^7-\frac{10!}{3!}$

(c) How many possibilities are there that have at least one repeated digit, and don't start with a 0 or a 1?

above we showed $\left(10^7-\frac{10!}{3!}\right)$ with at least one repeat. $\frac{8}{10}$ of these do not start with 0 or 1.

Total is $(0.8)\left(10^7-\frac{10!}{3!}\right)=7516160$

10c. Student Solution #1

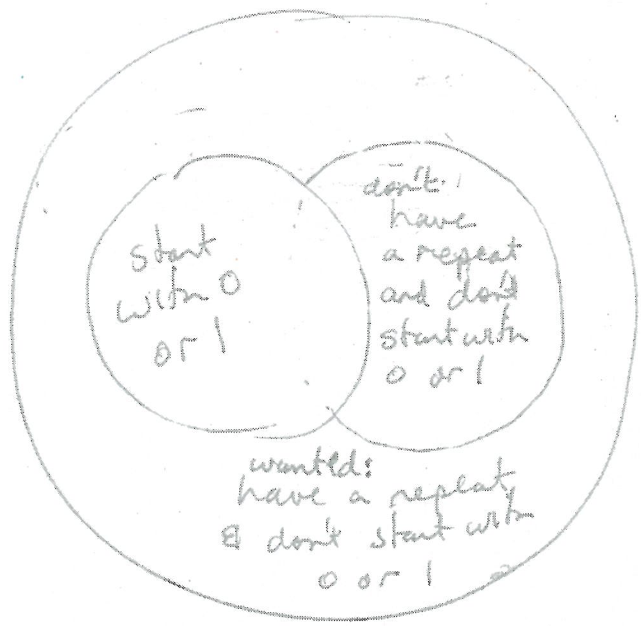
"at least one repeat & doesn't start with 0 or 1"

same as
at least one repeat take away repeats that start with 0 or 1

same as
at least one repeat take away all that start with 0 or 1 add back in ones with <u>no</u> repeat that start with 0 or 1

$$= \underbrace{\left(10^7 - \frac{10!}{3!}\right)}_{\text{at least one repeat}} - \underbrace{\underline{2}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}}_{\text{start with 0 or 1}} + \underbrace{2\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}_{\text{all different, start with 0 or 1}}$$

10c. Student Solution #2



have a repeat &
don't start with 0 or 1 means
all numbers − start with 0 or 1 − don't have a repeat and don't start with 0 or 1

$$10^7 - 2\cdot 10^6 - \frac{8\cdot 9!}{3!}$$