Math 2001, Activity Sheet for a Game (Take 2 or 1)

A two-player game for you to play

I want you to play the following game with a partner.

The board consists of n tokens. If you are working in person, use pencils, coins, torn up bits of paper, or whatever you have on hand. If you are working online, try an online whiteboard app like whiteboardfox.com, where you can draw some tokens (circles, perhaps) you can both see online.

At each turn, a player must take either 1 or 2 tokens off the board (which are discarded – place them aside if physical, or erase them, if on a whiteboard).

The player who takes the last token(s) on their turn, leaving 0 tokens, is the winner.

Example game beginning with 4 tokens:

- 4 tokens are on the board, and Alice takes 1.
- 3 tokens remain, and Bob takes 2.
- 1 token remains, so Alice takes it and wins.

Your task: Play the game, and try to figure out the winning strategy.

Analysing the Game: Winning and Losing Positions

Now that you are getting good at the game, we will do a little analysis. Consider the following definition:

Definition 1. *A board with n tokens is called a* winning position *if the first player to move can always win the game, no matter how his opponent moves. Otherwise it is called a* losing position.

In other words, n is a winning position if the first player, provided he is smart enough to make all the right moves, can always win, no matter what the second player does. Otherwise it is a losing position. (There are no ties in this game.)

Fill in a chart of winning and losing positions:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 . . .
win?																

What is the pattern of winning and losing positions? Fill in the blank in this theorem:

Theorem 1. *Let n be a positive integer. If _____, then n is a losing position. Otherwise n is a winning position.*

Analysing the Game: Writing a proof

Fill in the blank with (winning/losing) and write a proof of the theorem.

Theorem 2. *A board with* 2 *tokens is a* _____ *position.*

Proof.

□

Fill in the blank with (winning/losing) and write a proof of the theorem.

Theorem 3. *A board with* 3 *tokens is a* _____ *position.*

Proof.

□

Fill in the blank with (winning/losing) and write a proof of the theorem. (Hint: you can call on the previous Theorem.)

Theorem 4. *A board with* 4 *tokens is a* _____ *position.*

Proof.

□

Fill in the blank with (winning/losing) and write a proof of the theorem.

Theorem 5. *A board with* 6 *tokens is a* _____ *position.*

Proof.

□

Next I will ask you to prove the full theorem. The following are two true lemmas that may help you organize your thoughts.

Lemma 1. *If the first player can leave the second player with a losing position, then the current position is winning.*

Lemma 2. *If the first player has no choice but to leave the second player with a winning position, then the current position is losing.*

(Note: You can try to write proofs of these lemmas, if you like. Or you can think really hard about them and come to believe them. Nevertheless, they are true!)

Now, prove the full theorem. Give it your best shot to write a full, nice proof.

Theorem 6. *Let n be a positive integer. If* _____*, then n is a losing position. Otherwise n is a winning position.*