

## 1 The rules of $n \times n$ Simplified Sudoku

An  $n \times n$  square of boxes whose positions are filled with the numbers  $1, 2, \dots, n$  (one number per box) is called a *completed Sudoku*. The Sudoku is called *valid* if no row or column contains a repeated number. Otherwise it is called *invalid*.

Here is an example of a valid Sudoku:

1	3	2
3	2	1
2	1	3

Here is an example of an invalid Sudoku:

3	1	2
3	2	1
2	1	3

It is invalid because the first column contains a repeated 3. It is also invalid because the second column contains a repeated 1.

(In regular Sudoku puzzles, the box is  $9 \times 9$ , and the same no-repeat rule must apply to  $3 \times 3$  sub-boxes; here we do not use such a rule.)

## 2 Task

Prove that there is no way to complete the table below so that it is valid Sudoku. (In other words, this puzzle has no solution.)

1		3
2	1	

*Advice and thoughts: A proof is a written logical argument that would convince your peers of the truth of the statement, beyond any reasonable doubt. This is a writing assignment. Your proof may be only a few sentences long, but you should choose those sentences so as to be as clear as possible.*

## 3 Solutions and Discussion

### 3.1 A fairly nice solution

Let us label the columns  $C_1, C_2$  and  $C_3$ , from left to right. Similarly, the rows will be labelled  $R_1, R_2$  and  $R_3$  from top to bottom.

Since  $R_3$  contains a 1 and a 2 already, the rules of the game dictate that the third position of  $R_3$  can only be filled by a 3. But then  $C_3$  would contain a 3 in both the first and last positions. This is not allowed by the rules of the puzzle!

We have shown that any proposed solution that satisfies the rules for  $R_3$  will fail them for  $C_3$ , hence no valid solution exists.

### 3.2 Another decent solution

We will assume there is a valid solution, and reach a contradiction. Let us focus on the lower-right square in the grid, call it  $X$ . By the rules of the puzzle, the third row cannot contain any repeated entries. This third row already contains a 1 and a 2. Using the process of elimination, we conclude that  $X$  (the final position in this row) must contain a 3. But then the third column contains two 3's, a contradiction to the rules of the puzzle. We conclude that the puzzle has no solution.

### 3.3 Another example

We proceed by contradiction. Suppose that there is a valid solution to this simplified Sudoku, and label the columns of the puzzle  $C_i$  and the rows  $R_i$  with  $1 \leq i \leq 3$  and with labeling running from left to right and top to bottom respectively. The notation  $(i, j)$  will then denote the position at the intersection of  $R_i$  and  $C_j$ .

Since  $C_1$  contains a '1' at  $(1, 1)$  and a '2' at  $(3, 1)$ , then position  $(2, 1)$  must contain a '3'. Looking at the table, we can then see that  $R_1, R_2, C_1$ , and  $C_3$  all contain a '3', and since we need a '3' in every row and in every column, we must have a '3' in  $R_3$  and  $C_2$ ; in other words, there must be a '3' at position  $(3, 2)$ . However, there is already a '1' at position  $(3, 2)$ , so we have a contradiction. Thus, no valid solution to the given puzzle can exist.

### 3.4 Some useful strategies

1. If you're going to refer to something often, give it a memorable name.
2. Always be completely precise: 'a 3 in the row' doesn't say which column.
3. One logical step per sentence. Simple sentences.
4. Avoid redundancy.

### 3.5 ChatGPT: How did it do?

#### 3.5.1 Here is a prompt to ChatGPT:

Here is a 3x3 grid: row 1 is "1, blank, 3", row 2 is "blank, blank, blank", row 3 is "2, 1, blank". The goal is to fill every blank square with either 1, 2, or 3, so that there are no repeated numbers in any individual row or column. Prove that there is no solution.

#### 3.5.2 Its response is:

This is a classic problem in combinatorics and is known as a Latin square. To prove that there is no solution, we can use a proof by contradiction. We will assume that there is a solution and show that it leads to a contradiction.

Let's call the first column C1, the second column C2, and the third column C3. From the given information, we know that:

- Row 1 contains 1, blank, 3
- Row 2 contains blank, blank, blank
- Row 3 contains 2, 1, blank

We also know that there are no repeated numbers in any individual row or column.

Since C1 contains 1 in the first row, it cannot contain 1 in any other row. Therefore, C1 must contain 2 in the third row. But this means that C2 must contain 1, and C3 must contain 3.

Since C2 contains 1 in the second row, it cannot contain 1 in any other row. Therefore, C2 must contain 2 in the third row. But this means that C3 contains 1, which contradicts the assumption that C3 contains 3.

Therefore, there is no solution and the assumption that there is a solution leads to a contradiction.

### 3.6 Comments on ChatGPT

This is actually very nicely organized as far as the set up. ChatGPT is always a bit long-winded; you don't need to do quite this much work.

However, the logic part ('Since C1 contains 1..... the assumption that C3 contains 3.') is nonsense. A major issue is that ChatGPT doesn't seem to understand the difference between the hypotheses and the things it is trying to conclude. For example, it *concludes* C1 must contain 2 in the third row. But that's a *given* of the problem, not something to try to prove. It also contradicts itself, etc.

I would give ChatGPT a 0 on logic, and a 4 out of 4 on writing (I can't really complain about its writing, even though it's longwinded).