MATHEMATICS 2001 – CONTRADICTION PUZZLES!

It's puzzle time! The goal today is to produce some nicely-written proofs. The method of contradiction will work well for these, but it is not mandated – any proof will do.

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth." – Sherlock Holmes, in the novel The Sign of the Four (1890) by Sir Arthur Conan Doyle

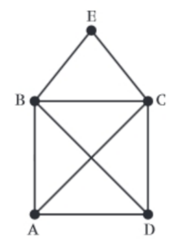
Theorem 1. *There is no largest integer.*

Theorem 2. *There is no smallest positive rational number.*

Theorem 3. *There are no integers a and b such that $4a + 18b = 1$.*

Theorem 4. *Suppose x is rational and y is irrational. Then xy is irrational.*

Theorem 5. *Consider the following picture:*



It is impossible to traverse this diagram along the edges in a loop (ending where you begin) using each edge exactly once*.*

Note: This is a finite thing to check, but that is tedious (you'd have to try all possible loops!). Instead, imagine there did exist a loop, and reach a contradiction.

Theorem 6. *There are infinitely many primes p such that $p + 2$ is composite.*

Interesting fact: Mathematicians believe there are also infinitely many primes p such that $p + 2$ is prime, but no one has been able to prove it.

Theorem 7. *Consider a unit square with sides of length 1. Suppose 5 points are placed inside this square. Then there exist two points x and y among these five, such that the distance between x and y is less than or equal to $1/\sqrt{2}$.*