

## MATHEMATICS 2001 – CONTRADICTION PUZZLES!

**It's puzzle time!** The goal today is to produce some nicely-written proofs. The method of contradiction will work well for these, but it is not mandated – any proof will do.

“When you have eliminated the impossible, whatever remains, however improbable, must be the truth.”  
– Sherlock Holmes, in the novel *The Sign of the Four* (1890) by Sir Arthur Conan Doyle

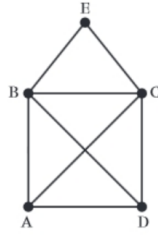
**Theorem 1.** *There is no largest integer.*

**Theorem 2.** *There is no smallest positive rational number.*

**Theorem 3.** *There are infinitely many primes  $p$  such that  $p + 2$  is composite.*

Interesting fact: Mathematicians believe there are also infinitely many primes  $p$  such that  $p + 2$  is prime, but no one has been able to prove it.

**Theorem 4.** Consider the following picture:



*It is impossible to traverse this diagram along the edges in a loop (ending where you begin) using each edge exactly once.*

Note: This is a finite thing to check, but that is tedious (you'd have to try all possible loops!). Instead, imagine there did exist a loop, and reach a contradiction.

**Theorem 5.** Consider a unit square with sides of length 1. Suppose 5 points are placed inside this square. Then there exist two points  $x$  and  $y$  among these five, such that the distance between  $x$  and  $y$  is less than or equal to  $1/\sqrt{2}$ .