

# Proof Test 6

Math 2001, Spring 2023. Katherine E. Stange.

**Theorem 1.** *Let  $n \geq 0$  be an integer. Then 3 divides  $5^{2n} - 1$ .*

*Hint: This is designed to be easiest using proof by induction.*

*Proof.* We will prove this by inducting on  $n$ .

**Base case:** Observe that 3 divides  $5^0 - 1 = 0$ .

**Inductive step:** Assume that the theorem holds for  $n = k \geq 0$ . We will prove that theorem holds for  $n = k + 1$ .

By the inductive assumption,  $5^{2k} - 1 = 3\ell$  for some integer  $\ell$ . We wish to use this to show that the quantity  $5^{2k+2} - 1$  is a multiple of 3. We will manipulate this quantity in order to express it in terms of the quantity  $5^{2k} - 1$ , at which point we can use the inductive hypothesis. Explicitly,

$$5^{2k+2} - 1 = 5^2 \cdot 5^{2k} - 1 = 5^2(5^{2k} - 1 + 1) - 1 = 5^2(3\ell + 1) - 1 = 75\ell + 24.$$

Since  $75\ell$  is a multiple of 3 and so is 24, we see that  $5^{2k+2} - 1$  is a multiple of 3. □

## Induction setup variation

Here are several variations. First, we might phrase the inductive setup as ‘strong induction’. The difference from the last proof is in bold.

*Proof.* We will prove this by inducting on  $n$ .

**Base case:** Observe that 3 divides  $5^0 - 1 = 0$ .

**Inductive step:** Assume that the theorem holds for  $n \leq k$ , where  $k \geq 0$ . We will prove that theorem holds for  $n = k + 1$ .

By the inductive assumption,  $5^{2k} - 1 = 3\ell$  for some integer  $\ell$ . We wish to use this to show that the quantity  $5^{2k+2} - 1$  is a multiple of 3. We will manipulate this quantity in order to express it in terms of the quantity  $5^{2k} - 1$ , at which point we can use the inductive hypothesis. Explicitly,

$$5^{2k+2} - 1 = 5^2 \cdot 5^{2k} - 1 = 5^2(5^{2k} - 1 + 1) - 1 = 5^2(3\ell + 1) - 1 = 75\ell + 24.$$

Since  $75\ell$  is a multiple of 3 and so is 24, we see that  $5^{2k+2} - 1$  is a multiple of 3. □

## Algebra variation 1

Another variation is in the way the algebra piece (the core of the proof) happens. Again, the difference is in bold.

*Proof.* We will prove this by inducting on  $n$ .

**Base case:** Observe that 3 divides  $5^0 - 1 = 0$ .

**Inductive step:** Assume that the theorem holds for  $n = k \geq 0$ . We will prove that theorem holds for  $n = k + 1$ .

By the inductive assumption,  $5^{2k} - 1 = 3\ell$  for some integer  $\ell$ . We wish to use this to show that the quantity  $5^{2k+2} - 1$  is a multiple of 3.

**Rearranging the inductive assumption,  $5^{2k} = 3\ell + 1$ . Multiplying by  $5^2$ , we have**

$$5^{2k+2} = 5^2(3\ell + 1) = 75\ell + 25$$

**Therefore**

$$5^{2k+2} - 1 = 75\ell + 24.$$

Since  $75\ell$  is a multiple of 3 and so is 24, we see that  $5^{2k+2} - 1$  is a multiple of 3. □

## Algebra variation 2

Another algebra variation in bold.

*Proof.* We will prove this by inducting on  $n$ .

**Base case:** Observe that 3 divides  $5^0 - 1 = 0$ .

**Inductive step:** Assume that the theorem holds for  $n = k \geq 0$ . We will prove that theorem holds for  $n = k + 1$ .

By the inductive assumption,  $5^{2k} - 1 = 3\ell$  for some integer  $\ell$ . We wish to use this to show that the quantity  $5^{2k+2} - 1$  is a multiple of 3.

**We rearrange the inductive assumption to obtain**

$$5^{2k} = 3\ell + 1.$$

**Now write  $5^{2k+2} - 1 = 3a$ . Our goal is to show that  $a$  is an integer. We will solve for  $a$ . We have**

$$3a = 5^{2k+2} - 1 = 25(5^{2k}) - 1 = 25(3\ell + 1) - 1 = 75\ell + 24$$

so

$$a = 25\ell + 8$$

**which is an integer. Therefore  $5^{2k+2} - 1$  is a multiple of 3.** □

## ChatGPT

For each of the AI answers, try to discern if it is correct or wrong.

We will prove this statement by mathematical induction on  $n$ .

Base case:  $n = 0$ .

In this case,  $5^{2n} - 1 = 5^0 - 1 = 0$ , which is divisible by 3. So the statement is true for  $n = 0$ .

Inductive step:

Assume that the statement is true for some arbitrary integer  $k \geq 0$ , that is, 3 divides  $5^{2k} - 1$ . We will show that the statement is also true for  $k + 1$ , that is, 3 divides  $5^{2(k+1)} - 1$ .

We have:

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^{2k} \cdot 5^2 - 1 \\ &= 25 \cdot 5^{2k} - 1 \\ &= (24 + 1) \cdot 5^{2k} - 1 \\ &= 24 \cdot 5^{2k} + 5^{2k} - 1 \\ &= 3 \cdot 8 \cdot 5^{2k} + 5^{2k} - 1 \\ &= 3 \cdot (8 \cdot 5^{2k} + 5^{2k} - 1) \end{aligned}$$

Since  $8 \cdot 5^{2k} + 5^{2k} - 1$  is an integer, it follows that 3 divides  $5^{2(k+1)} - 1$ , as required.

Therefore, by mathematical induction, the statement is true for all non-negative integers  $n$ .

Error in the last line of algebra.

## Bing

To prove that 3 divides  $5^{(2n)}-1$ , we can use mathematical induction.

First, we can check that the statement holds for  $n=0$ :  $5^{(2*0)}-1=24$  which is divisible by 3.

Next, we assume that the statement holds for some integer  $k \geq 0$  and try to prove it for  $k+1$ .

We have:

$$\begin{aligned}5^{(2(k+1))}-1 &= 25 * 5^{(2k)} - 1 = (24 + 1) * 5^{(2k)} - 1 \\&= (3 * 8) * 5^{(2k)} + (5^{(2k)} - 1) \\&= 3 * (8 * 5^{(2k)}) + (5^{(2k)} - 1)\end{aligned}$$

Since we know that 3 divides  $(5^{(2k)}-1)$ , we can write:

$$\begin{aligned}&= 3m + 3 * (8 * 5^{(2k)}) \\&= 3(m + (8 * 5^{(2k)}))\end{aligned}$$

where  $m$  is some integer.

Therefore, we have shown that if the statement holds for some integer  $k \geq 0$ , then it also holds for  $k+1$ .  
By mathematical induction, we conclude that the statement holds for all integers  $n \geq 0$ .

Hence, we have proved that 3 divides  $5^{(2n)}-1$  for all integers  $n \geq 0$ .

This actually looks correct! The algebra slightly is unusual (not the way I mostly saw students doing it) but correct.

## Bard

We can prove this by mathematical induction.

**Base case:** When  $n = 0$ , we have  $5^{2 \cdot 0} - 1 = 1 - 1 = 0$ , which is divisible by 3.

**Inductive step:** Assume that 3 divides  $5^{2k} - 1$  for some integer  $k \geq 0$ . We want to show that 3 divides  $5^{2(k+1)} - 1$ .

We have  $\begin{aligned}$

$$5^{2(k+1)}-1 = 5^{2k} \cdot 5^2 - 1$$

$$= (5^{2k}-1) + 5^2$$

$\text{\textit{(by the distributive property)}}$

$\text{\textit{(by the inductive hypothesis)}}$

$\text{\textit{(since 3 divides } 5^2)}$

$$\equiv 0 \pmod{3}.$$

$\end{aligned}$

Therefore, 3 divides  $5^{2(k+1)} - 1$  for all integers  $k \geq 0$ .

**Conclusion:** By the principle of mathematical induction, 3 divides  $5^{2n} - 1$  for all integers  $n \geq 0$ .

Bard provided some of its answer in LaTeX, so the equational part is actually (after fixing the LaTeX compile errors):

$$\begin{aligned}
5^{2(k+1)} - 1 &= 5^{2k} \cdot 5^2 - 1 \\
&= (5^{2k} - 1) + 5^2 \\
&\text{(by the distributive property)} \\
&\text{(by the inductive hypothesis)} \\
&\text{(since 3 divides } 5^2) \\
&\equiv 0 \pmod{3}.
\end{aligned}$$

This algebra is a mess. And no, 3 does not divide  $5^2$ .  
Here's another draft:

We can prove this by mathematical induction.

**Base case:** When  $n = 0$ , we have  $5^{2n} - 1 = 5^0 - 1 = 1 - 1 = 0$ , which is divisible by 3.

**Inductive step:** Assume that 3 divides  $5^{2k} - 1$  for some integer  $k \geq 0$ . We want to show that 3 divides  $5^{2(k+1)} - 1$ .

We have

$$5^{2(k+1)} - 1 = 5^2 \cdot 5^{2k} - 1 = (5^2 - 1) \cdot 5^{2k} + 5^{2k} - 1 = 4 \cdot 5^{2k} - 1.$$

Since 3 divides  $5^{2k} - 1$ , it also divides  $4 \cdot 5^{2k} - 1$ , since 4 is a multiple of 3. Therefore, 3 divides  $5^{2(k+1)} - 1$ .

This completes the inductive step. By the principle of mathematical induction, 3 divides  $5^{2n} - 1$  for all integers  $n \geq 0$ .

The algebra devolves here again. And no, 4 is not a multiple of 3.

We can prove this by mathematical induction.

**Base case:** When  $n = 0$ , we have  $5^{2 \cdot 0} - 1 = 1 - 1 = 0$ , which is divisible by 3.

**Inductive step:** Assume that 3 divides  $5^{2k} - 1$  for some integer  $k \geq 0$ . We want to show that 3 divides  $5^{2(k+1)} - 1$ .

We can write  $5^{2(k+1)} - 1$  as follows:

$$5^{2(k+1)} - 1 = 5^{2k} \cdot 5^2 - 1 = (5^{2k} - 1) \cdot 5^2 + 5^2$$

By the inductive hypothesis, 3 divides  $5^{2k} - 1$ . Therefore, 3 divides  $(5^{2k} - 1) \cdot 5^2$ .

Also, 3 divides  $5^2$ , since 5 is a multiple of 3.

Therefore, 3 divides  $(5^{2k} - 1) \cdot 5^2 + 5^2$ , which is equal to  $5^{2(k+1)} - 1$ .

This completes the inductive step. By the principle of mathematical induction, 3 divides  $5^{2n} - 1$  for all integers  $n \geq 0$ .

Again, 3 does not divide  $5^2$ . It is making this up to excuse an algebra error at the last equal sign.

In conclusion, the AIs are very good at setting up induction but can't get algebra correct, and, interestingly, repeatedly go on to assert total falsehoods to make up for algebra errors. This is something I see with the AIs a lot – they appear to take whatever they have already written as truth and make up whatever falsehoods are necessary to justify it after the fact.