

# Proof Test 5

Math 2001, Spring 2023. Katherine E. Stange.

**Theorem 1.** *Let  $x, y \in \mathbb{R}$ . If  $x^3 + x^2 - y - 1 \neq 0$  then  $x \neq 1$  or  $y \neq 1$ .*

*Hint: This is designed to be easiest using proof by contrapositive.*

*Proof by contrapositive.* We will prove this by contrapositive. Let  $x = 1$  and  $y = 1$ . Then we may compute

$$x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 2 - 2 = 0.$$

□

*Proof by contradiction.* We will prove this by contradiction. Assume that  $x^3 + x^2 - y - 1 \neq 0$  but  $x = 1$  and  $y = 1$ . Then we may compute

$$x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 2 - 2 = 0.$$

This is a contradiction to the assumption that  $x^3 + x^2 - y - 1 \neq 0$ .

□

Both of these are fine, but notice how contrapositive is cleaner than contradiction, while having the same essential ‘core.’

## Common error: stating what you are required to prove

*Proof with an error #1.* We will prove this by contrapositive. **If  $x = 1$  and  $y = 1$ , then  $x^3 + x^2 - y - 1 = 0$ .** Let  $x = 1$  and  $y = 1$ . Then we may compute

$$x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 2 - 2 = 0.$$

□

In the proof above, the bolded statement “If  $x = 1$  and  $y = 1$ , then  $x^3 + x^2 - y - 1 = 0$ .” is stated as if it is a known fact. But we don’t know this yet, as we have not proven it yet. Here’s how to fix it, by making clear that we are talking *about* this statement, not *asserting it is true*.

*Proof with an error #1 – fixed one way.* We will prove this by contrapositive. **In other words, we will prove that if  $x = 1$  and  $y = 1$ , then  $x^3 + x^2 - y - 1 = 0$ .** Let  $x = 1$  and  $y = 1$ . Then we may compute

$$x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 2 - 2 = 0.$$

□

*Proof with an error #1 – fixed another way.* We will prove this by contrapositive. **The contrapositive statement is “if  $x = 1$  and  $y = 1$ , then  $x^3 + x^2 - y - 1 = 0$ .”** Let  $x = 1$  and  $y = 1$ . Then we may compute

$$x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 2 - 2 = 0.$$

□

## Common error: proof by left-side-right-side

*Proof with an error #2 example 1.* We will prove this by contrapositive. Let  $x = 1$  and  $y = 1$ . Then

$$\begin{aligned}x^3 + x^2 - y - 1 &= 0 \\1^3 + 1^2 - 1 - 1 &= 0 \\2 - 2 &= 0 \\0 &= 0 \quad \checkmark\end{aligned}$$

□

In this proof the writer wanted to verify that  $x^3 + x^2 - y - 1 = 0$  but instead of calculating it directly as in the model proofs, the writer did a right-side-left-side proof.

*Proof with an error #2 example 2.* We will prove this by contrapositive. Let  $x = 1$  and  $y = 1$ . **Then**  $x^3 + x^2 - y - 1 = 0, 0 = 0$ . □

In the proof above, the left-side-right-side proof is written into a sentence, but it just reads like someone tacked on a  $0 = 0$  at the end of the sentence for no reason.

I have a whole separate document about why left-side-right-side is not a reasonable, logically valid way to write up a proof (see the website), but in short, the problem is that you write  $x^3 + x^2 - y - 1 = 0$  as a statement before it has been proven. Then you do some work and conclude that  $0 = 0$ , a fact which everyone knew long before. So the logic is ‘backwards.’ It also obscures possible logical errors (as demonstrated in my separate document).

## Bing’s response

I can prove it using a proof by contradiction. Suppose that  $x = 1$  and  $y = 1$ , then we have:

$$(1)^3 + (1)^2 - (1) - 1 = 0$$

This contradicts the statement “ $x^3 + x^2 - y - 1$  is not equal to 0”. Therefore, if  $x^3 + x^2 - y - 1$  is not equal to 0, then  $x$  is not equal to 1 or  $y$  is not equal to 1.

I hope this helps! Let me know if you have any other questions.

This is totally fine.

Sure! Here’s another proof:

Suppose that  $x^3 + x^2 - y - 1 = 0$  and  $x = 1$ , then we have:

$$(1)^3 + (1)^2 - (y) - 1 = 0$$

$$y = -1$$

Therefore, if  $x^3 + x^2 - y - 1$  is not equal to 0, then  $x$  is not equal to 1 or  $y$  is not equal to 1.

I hope this helps! Let me know if you have any other questions.

Bizarrely, it tries to prove that if the polynomial IS zero, then either  $x$  is not 1 or  $y$  is not 1. It makes a sign error that appears to make this true. But this is not what we were trying to prove anyway.

## ChatGPT's response



We will prove this statement using a proof by contradiction.



Assume that  $x^3 + x^2 - y - 1$  is not equal to 0 and that  $x = 1$  and  $y = 1$ .

Then we have:

$$1^3 + 1^2 - 1 - 1 = 0$$

This contradicts the assumption that  $x^3 + x^2 - y - 1$  is not equal to 0.

Therefore, we have shown that if  $x^3 + x^2 - y - 1$  is not equal to 0, then  $x$  is not equal to 1 or  $y$  is not equal to 1.

This is totally fine.



Yes, we can also prove this statement using contrapositive.



The contrapositive of the statement "If  $x^3 + x^2 - y - 1$  is not equal to 0, then  $x$  is not equal to 1 or  $y$  is not equal to 1" is "If  $x = 1$  or  $y = 1$ , then  $x^3 + x^2 - y - 1 = 0$ ."

Now, we will show that the contrapositive statement is true.

If  $x = 1$ , then  $x^3 + x^2 - y - 1 = 1^3 + 1^2 - y - 1 = 0 - y = -y$ , which can be equal to 0 if and only if  $y = 0$ .

Similarly, if  $y = 1$ , then  $x^3 + x^2 - y - 1 = x^3 + x^2 - 2 = (x - 1)(x^2 + x + 2)$ . Since  $x^2 + x + 2$  is always positive (as its discriminant is negative), we can conclude that  $x^3 + x^2 - y - 1$  can be equal to 0 if and only if  $x = 1$ .

Therefore, we have shown that the contrapositive statement is true, and hence the original statement "If  $x^3 + x^2 - y - 1$  is not equal to 0, then  $x$  is not equal to 1 or  $y$  is not equal to 1" is also true.

It gets the contrapositive wrong. The contrapositive it aims to prove is not a true statement. It then does cases (which makes sense given the hypotheses) but shows that the polynomial can be zero, not that it must be zero.

## Bard's response

Bard provided three drafts. Bard was by far the most creative.

◆ Sure. We can prove this by contradiction. Suppose that  $x = 1$  and  $y = 1$ . Then,  $x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 0$ . This contradicts our assumption that  $x^3 + x^2 - y - 1 \neq 0$ . Therefore, we must have either  $x \neq 1$  or  $y \neq 1$ .

Alternatively, we can use the Factor Theorem to prove this statement. The Factor Theorem states that if  $f(x)$  is a polynomial with real coefficients and  $x = a$  is a root of  $f(x)$ , then  $(x - a)$  is a factor of  $f(x)$ . In this case, we have  $f(x) = x^3 + x^2 - y - 1$ . If  $x = 1$  is a root of  $f(x)$ , then  $(x - 1)$  is a factor of  $f(x)$ . This means that  $f(x)$  can be written in the form  $f(x) = (x - 1)g(x)$  for some polynomial  $g(x)$ . Substituting  $x = 1$  into this equation, we get  $(1 - 1)g(1) = 0$ . This implies that  $g(1) = 0$ . Therefore,  $y = g(1)$  must be equal to 0. This contradicts our assumption that  $y \neq 1$ . Therefore,  $x \neq 1$ .

Similarly, if we assume that  $y = 1$ , then we can use the Factor Theorem to conclude that  $x = 1$ . This contradicts our assumption that  $x \neq 1$ . Therefore, we must have either  $x \neq 1$  or  $y \neq 1$ .



The first of these two is ok except it gets 1 instead of 0 for the evaluated polynomial. It then proceeds as if it got 0.

The second of these two is totally weird. It assumes  $x = 1$  is a root of  $f(x)$ , and gets a contradiction, to conclude  $x \neq 1$ . Those two are not negations of one another. Also it uses the factor theorem incorrectly (it doesn't say anything about  $g(1)$  and it claims this shows  $y = g(1) = 1$  which is at this point just gibberish.

◆ Sure. We can prove this by contradiction. Suppose that  $x = 1$  and  $y = 1$ . Then,  $x^3 + x^2 - y - 1 = 1^3 + 1^2 - 1 - 1 = 0$ . This contradicts our assumption that  $x^3 + x^2 - y - 1$  is not equal to 0. Therefore,  $x$  must not equal 1 or  $y$  must not equal 1.

Here is another way to prove this:

Let  $f(x, y) = x^3 + x^2 - y - 1$ . We can see that  $f(1, 1) = 0$ . Therefore, by the Intermediate Value Theorem, there exists a point  $(x_0, y_0)$  such that  $f(x_0, y_0) = 0$  and  $x_0 \neq 1$  or  $y_0 \neq 1$ . This proves that  $x$  must not equal 1 or  $y$  must not equal 1.



The first one is fine now.

The second of these two is just pure crazy talk. The intermediate value theorem doesn't tell you anything in this context, and the existence of an  $x_0$  and  $y_0$  with the stated property doesn't prove what you are trying to prove...

[View other drafts](#) ▼

◆ Sure, I can help you with that.

We are given that  $x^3 + x^2 - y - 1 \neq 0$ . We want to show that  $x \neq 1$  or  $y \neq 1$ .

If  $x = 1$ , then  $x^3 + x^2 - y - 1 = 1 + 1 - y - 1 = -y$ . Since we are given that  $x^3 + x^2 - y - 1 \neq 0$ , then  $-y \neq 0$ , which implies that  $y \neq 0$ .

If  $y = 1$ , then  $x^3 + x^2 - 1 - 1 = x^2 - 1$ . Since we are given that  $x^3 + x^2 - y - 1 \neq 0$ , then  $x^2 - 1 \neq 0$ , which implies that  $x \neq 1$ .

Therefore, we have shown that if  $x^3 + x^2 - y - 1 \neq 0$ , then  $x \neq 1$  or  $y \neq 1$ .



This proof is interesting. It seems to have a big-scale plan that makes sense: if  $x = 1$ , then  $y \neq 1$  and vice versa. But it gets the algebra wrong in both cases, and the conclusion wrong in one case, and stumbles on anyway.