Proof Test 4

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Tools

You might find it useful to call on these basic facts and definitions.

Definition 1. *A number $x \in \mathbb{R}$ is* rational *if it can be written as $x = a/b$ where $a, b \in \mathbb{Z}$. Otherwise x is called* irrational.

Proposition 1. *The number $\sqrt{2}$ is irrational.*

Proposition 2. *A real number x is rational if and only if $-x$ is rational.*

Proposition 3. *Suppose $x = a + b$. If a and b are rational, then so is x.*

Proposition 4. *Suppose $x = a + b$. If a is rational and b is irrational, then x is irrational.*

Task

Please prove the following theorem.

Theorem 1. *Any rational number can be written as a sum of two irrational numbers.*

Hint: For example, $3 = \sqrt{2} + (3 - \sqrt{2})$.

Proof. Let x be a rational number. By Proposiion 1, $\sqrt{2}$ is irrational. By Proposition 2, $-\sqrt{2}$ is irrational. By Proposition 4, $x - \sqrt{2}$ is a sum of a rational and an irrational, hence irrational. Now, write

$$x = \sqrt{2} + (x - \sqrt{2}).$$

As we have seen above, $\sqrt{2}$ and $x - \sqrt{2}$ are irrational, so we have written x as a sum of two irrational numbers. $\square$

1 ChatGPT contradicts itself

It is not true that any rational number can be written as a sum of two irrational numbers. For example, the rational number 1 cannot be written as the sum of two irrational numbers.

To see why, suppose that 1 could be written as the sum of two irrational numbers x and y. Then we would have:

1 = x + y

Rearranging this equation, we get:

x = 1 - y

Now, suppose that x and y are both irrational. Then 1 - y must also be irrational, since the difference between two irrational numbers is not guaranteed to be rational. But this would imply that x is the sum of two irrational numbers (1 and -y), which is a contradiction. Therefore, it is not possible to write 1 as the sum of two irrational numbers.

However, it is true that any rational number can be written as the difference of two irrational numbers. To see why, let r be a rational number, and let a be an irrational number. Then r - a is also irrational (since the difference of a rational number and an irrational number is always irrational). So we have:

r = (r - a) + a

where (r - a) and a are both irrational. Therefore, any rational number can be written as the difference of two irrational numbers.