

Proof Test 2

Math 2001, Spring 2023. Katherine E. Stange.

Provide a proof of the following theorem.

Two sets are called *disjoint* if there are no elements that are common to both. Informally, this means they do not overlap. More formally, sets X and Y are *disjoint* if there is no $x \in X$ which also satisfies $x \in Y$.

Theorem 1. *Let X be a set of n elements. The number of ways to choose an ordered pair of two disjoint subsets of X is 3^n .*

Hint: To clarify the phrase ‘ordered pair’ by an example, if $X = \{1, 2, 3\}$, then the pair of subsets $\{1\}, \{2\}$ constitutes a different solution than the pair of subsets $\{2\}, \{1\}$. In other words, there is a first subset and a second subset.

1 A longer solution

Let X be a set of n elements. We wish to form two disjoint subsets $A, B \subseteq X$. For each element $x \in X$, we have three options:

1. place x in A ;
2. place x in B ;
3. do not place x in either A or B .

It is not possible to place x in both A and B , since they are disjoint. Thus for each element, there are three options.

The placement of each element is independent. Placing all the elements determines the two sets A and B , which will then be disjoint by construction, as required. There is no overcounting, since any difference in the placements will result in a different pair of sets A and B .

Therefore, by multiplication principle, there are 3^n ways to form the sets A and B .

2 A shorter solution

Call the two disjoint subsets A and B . To determine A and B , it is sufficient to place each element of X in one of three ways: in set A , in set B , or in neither set. The resulting pair will be disjoint. By the multiplication principle, there are 3^n ways to form the pair of subsets.

3 A proof using binomial coefficients

This proof is due to Mohammed.

Let A be a set of n elements. We will break the count into cases according to the size of the first set:

size	possible 1st set	possible second set
0	$\binom{n}{0}$	$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$
1	$\binom{n}{1}$	$\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{n-1}$
2	$\binom{n}{2}$	$\binom{n-2}{0} + \binom{n-2}{1} + \dots + \binom{n-2}{n-2}$
\vdots	\vdots	\vdots
$n-1$	$\binom{n}{n-1}$	$\binom{1}{0} + \binom{1}{1}$
n	$\binom{n}{n}$	$\binom{0}{0}$

By the multiplication principle, we multiply the second and third entries in each row, to obtain the number of possibilities of the given size. By the addition principles, we add all the results to obtain the total number of

possibilities. This gives

$$\begin{aligned} & \binom{n}{0} \left(\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} \right) \\ & + \binom{n}{1} \left(\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{n-1} \right) \\ & + \binom{n}{2} \left(\binom{n-2}{0} + \binom{n-2}{1} + \dots + \binom{n-2}{n-2} \right) \\ & + \dots \\ & + \binom{n}{n-1} \left(\binom{1}{0} + \binom{1}{1} \right) \\ & + \binom{n}{n} \binom{0}{0}. \end{aligned}$$

Next, we apply the theorem from class that

$$\sum_{i=0}^k \binom{k}{i} = 2^k$$

to simplify the result. We obtain

$$\binom{n}{0} 2^n + \binom{n}{1} 2^{n-1} + \binom{n}{2} 2^{n-2} + \dots + \binom{n}{n-1} 2 + \binom{n}{n} 2^0.$$

This is a binomial expansion of the quantity

$$(1 + 2)^n = 3^n.$$

4 ChatGPT

Let X be a set of n elements. We want to count the number of ways to choose an ordered pair of disjoint subsets of X .

Consider an element x of X . There are three possibilities:

x is in the first subset but not the second.

x is in the second subset but not the first.

x is in neither subset.

For each element x , there are three choices, so there are 3^n ways to choose the subsets.

To see this, we can think of constructing the ordered pair of subsets as follows: for each element of X , we choose whether to include it in the first subset, the second subset, or neither. There are 3 choices for each element, so there are 3^n possible choices in total.

Note that since the subsets are ordered, (A,B) is a different pair than (B,A) if A and B are different subsets. This is why we count the number of ordered pairs rather than the number of unordered pairs.

Therefore, there are 3^n ways to choose an ordered pair of disjoint subsets of X .

Discussion about ChatGPT's solution: If it deleted the last three paragraphs, it would be totally fine. It got a bit chatty toward the end and repeated itself. Also its discussion of ordered pairs is a bit weird. Its task is not to determine to why I asked it to consider ordered pairs instead of unordered ones, but it sounds like that's what it wants to do. But the first part is spot on.