Proof Test 1

Math 2001, Spring 2023. Katherine E. Stange.

Consider the following sets:

$$A = \{2x + 1 : x \in \mathbb{Z}\}$$
$$B = \{y^2 : y \in A\}$$

Prove that $B \subseteq A$.

1 A solution that can be improved

Solution:
The set B is $y^2 : y \in A \to$ it is the square of the set A.
The definition of A is $2x + 1$.
Since $A = 2x + 1$, and x is all integers, we have $y \in 2x + 1$.
Suppose $y^2 = (2x + 1)^2 = 2(2x^2 + 2x) + 1$.
But $2x + 1 : x \in \mathbb{Z}$ is an odd number.
$\to y^2$ is odd.
So $B \subseteq A$.

2 The same proof with some improvements

Solution:
The set B is $y^2 : y \in A \to$ it is the square of the set A.
The set B consists of y^2 where $y \in A$. Let y^2 be such an element.
The definition of A is $2x + 1$.
The set A consists of all integers of the form $2x + 1$ where x is an integer.
Since $A = 2x + 1$, and x is all integers, we have $y \in 2x + 1$.
Thus we can write $y = 2x + 1$ for some integer x.
Suppose $y^2 = (2x + 1)^2 = 2(2x^2 + 2x) + 1$.
Then $y^2 = (2x + 1)^2 = 2(2x^2 + 2x) + 1$.
But $2x + 1 : x \in \mathbb{Z}$ is an odd number.
But this expression has the form of an odd number.
$\to y^2$ is odd.
Therefore y^2 is odd.
And $B \subseteq A$.
Therefore $B \subseteq A$.

3 Some proof principles

1. **Notation:** Don't use symbols outside of math expressions (e.g. $\to$ for 'implies,' three dots for 'therefore,' or colon for 'such that'). Symbols can get mixed up with expressions, reducing readability. Use english words instead.

2. **Math grammar:** This is when math objects aren't used correctly in their grammatical roles. For example,

   (a) A set can't be equal to a number, e.g. 'the set A is $2x + 1$'.

   (b) Similarly, 'x is all integers' mistakes a quantity (one thing, x) for a plurality (a collection of things, 'all integers').

   (c) 'The definition of A is $2x + 1$': a definition can't 'be' a quantity.

The reader may be able to guess what is actually meant, but it is best to avoid making the reader work harder. Sometimes there may be more than one reasonable guess.

3. **Introducing variables:** We must always introduce each variable that is used before we conclude anything about it. So we have to 'let' or 'suppose' or 'where' or 'for' a variable into existence before we reason about it. Here are some ways to create a variable (in this case x) in your shared space with the reader:

   (a) 'Let x be an integer.' This means: Dear reader, please imagine an integer and call it x.

   (b) 'Suppose x is an integer.' This means: Dear reader, please imagine an integer and call it x.

   (c) 'Then $y = 2x + 1$ where x is an integer.' This means: Dear reader, we know (for some reasons earlier in the proof) that there's an integer which, when doubled and incremented, gives y, so let's think about this integer and call it x.

   (d) 'Then $y = 2x + 1$ for some integer x.' This means: Dear reader, we know (for some reasons earlier in the proof) that there's an integer which, when doubled and incremented, gives y, so let's think about this integer and call it x.

4. **Assumptions versus implications:**

   (a) An assumption is something we are allowed to assume (maybe because it is in the setup ('hypotheses') of the theorem, or because we are directing the reader to make a free choice. It's usually good to call out where an assumption comes from. For example:

      i. 'Suppose x is odd.' This means: Dear reader, we don't have any reason that x *needs* to be odd, or that logic dictates that it must be, but nevertheless, I want you to do the thought experiment that it is odd for now. (Perhaps this is one of several cases, or maybe we need to prove something about odd numbers, so we will start with an odd number.)

   (b) An implication is something that follows logically from what came before. We might say 'thus', 'therefore' or 'this implies'.

   (c) The word 'and' is problematic when used to introduce a new fact. Usually if you are using it, you could pick something that is more clearly in one of the two categories above. I think usually you mean to indicate an implication, but it has this vibe that you are bringing in something from left-field that the reader hasn't thought about yet, like a useful fact from previous knowledge – in which case the word 'Recall' is a nice word. I would be cautious with it.

5. **Set vs. element:** Don't mix up $\subseteq$ and $\in$.

4 Model Solution 1

We must show that every element of B is an element of A. Let $b \in B$. Then $b = y^2$ for some $y \in A$. Since $y \in A$, $y = 2x + 1$ for some $x \in \mathbb{Z}$. Thus $b = y^2 = (2x + 1)^2 = 4x^2 + 4x + 1 = 2(2x^2 + 2x) + 1$. Since $2x^2 + 2x \in \mathbb{Z}$, this shows that $b \in A$.

5 Model Solution 2

The set A is the set of odd integers. The set B consists of squares of odd integers. The square of an odd integer is odd. Thus every element of B is odd. In other words, $B \subseteq A$.