Proof Quiz #12 – Solution

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

YES / NO

Please write your best written proof of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *Let T be a tree with at least one vertex. Then there exists a way to label the vertices of T by 0s and 1s (each vertex is labelled by one number, repeats allowed), so that every edge joins a 0 to a 1 (in other words, there are no edges connecting a 0 to a 0 or a 1 to a 1).*

Hint: You may wish to give this property a convenient name, like “label-able.” Then you must prove that all trees are label-able. To help gain familiarity with this idea, notice that a triangle graph (a complete graph on 3 vertices) cannot be labelled in this way. But a square graph (a cycle with 4 vertices) can (just label so diagonal corners agree).

Proof. We will induct on the number of vertices.

Base Case: Suppose T is a tree with one vertex. Then it has no edges, and so we can label it with '0' and satisfy the indicated requirement. Therefore any tree with one vertex is labellable.

Inductive Step: Let $n > 1$. Suppose all trees with k vertices, for any $1 \leq k < n$, are labellable. We will show that any tree with n vertices is labellable.

Let T be a tree with n vertices. We will show how to label it.

The tree T has a leaf, call it v. Then $T - v$ is a tree with $n - 1$ vertices, and hence (by the inductive hypothesis), is labellable. Choose such a labelling. Then v, as a leaf, is adjacent to exactly one vertex of $T - v$. Label v with the opposite label to that of the adjacent vertex. This results in a labelling of T. It is a valid labelling because every edge in $T - v$ satisfies the required condition by the inductive hypothesis, while the edge adjacent to v satisfies the required condition by construction.

Therefore T is labellable. □