Proof Quiz #11 – Solution

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

YES / *NO*

Please write your best written proof of the following theorem. You will be graded on logic as well as writing. This proof requires a definition.

Definition 1. *Let $x \in \mathbb{R}$. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let $\mathrm{frac}(x)$ denotes $x - \lfloor x \rfloor$.*

For example, $\lfloor 17.6 \rfloor = 17$ and $\mathrm{frac}(17.6) = 0.6$. These are usually called the *integer part* and *fractional part*, respectively.

Theorem 1. *Let $\alpha \in \mathbb{R}$. Then, among the values*

$$0, \mathrm{frac}(\alpha), \mathrm{frac}(2\alpha), \ldots, \mathrm{frac}(n\alpha)$$

there are two which differ by at most $1/n$.

Hint: Please be careful, if applicable, about open and closed intervals.

Proof by pigeonhole principle. Divide the half-open unit interval $[0, 1)$ into n disjoint subintervals:

$$[0, 1/n), [1/n, 2/n), \ldots, [(n-1)/n, 1).$$

The union of these subintervals is $[0, 1)$.

Consider the $n+1$ quantities 0, α, 2α, …, $n\alpha$. Taking the fractional part of these results in $n+1$ quantities

$$0, \mathrm{frac}(\alpha), \mathrm{frac}(2\alpha), \ldots, \mathrm{frac}(n\alpha)$$

which lie in $[0, 1)$. Assign each of these $n+1$ quantities to the corresponding subinterval in which it lies. This places $n+1$ quantities into n intervals, so by the pigeonhole principle, there must be some interval containing at least two of them. In particular, there is some integer k so that

$$\mathrm{frac}(i\alpha), \mathrm{frac}(j\alpha) \in [k/n, (k+1)/n).$$

So,

$$|\mathrm{frac}(i\alpha) - \mathrm{frac}(j\alpha)| < 1/n.$$

□

By contradiction. Suppose, for a contradiction, that no two of the $n+1$ quantities

$$0, \mathrm{frac}(\alpha), \mathrm{frac}(2\alpha), \ldots, \mathrm{frac}(n\alpha)$$

lie within a distance of $1/n$. Then, in particular they are distinct. They all lie in the interval $[0,1)$ by definition. Relabel them in order of size so that

$$0 < x_1 < \ldots < x_n < 1.$$

Then,

$$x_{i+1} - x_i > 1/n.$$

Therefore,

$$x_n = x_n - 0 = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - 0) > 1/n + 1/n + \cdots + 1/n = 1.$$

This is a contradiction. □