Proof Quiz #4 Solution

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

$$YES \quad / \quad NO$$

Please write your best written **combinatorial proof** of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Important: I ask for a **combinatorial proof**, as we defined in class and Section 3.10 of the textbook (a combinatorial proof is one which proves an equation by showing both sides are correct answers to the same counting problem). Other methods of proof won't count for this quiz.

Hint: Consider chaired committees.

Proof. Let $n \geq 1$. Consider the problem of counting the number of ways to choose a non-empty committee from n people, such that one of its members is designated the chair.

One way to count such committees is to consider each size as a separate case. There are $\binom{n}{k}$ committees of size k. For each such a committee, there are k ways to designate a chair. Hence there are $k\binom{n}{k}$ chaired committees of size k. As the sizes can range from $k = 1$ to $k = n$, the total is

$$\sum_{k=1}^{n} k\binom{n}{k} = \sum_{k=0}^{n} k\binom{n}{k}.$$

(Notice that beginning the index at $k = 0$ instead of $k = 1$ makes no difference to the quantity, since $0\binom{n}{0} = 0$.)

A different way to count such committees is by an application of the multiplication principle, as follows. First, choose a chair, which can be done in n different ways. Then, having chosen a chair, choose a subset of the remaining people to form the rest of the committee. There are 2^{n-1} possible subsets of the $n - 1$ remaining people. The result is therefore $n2^{n-1}$.

Since both of these are valid methods to answer the same counting problem, the answers must be equal. □