Proof Quiz #2

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Honor Code Rules

Proof Quizzes are open book, but are to be completed on your own without collaboration. To be specific, you may use your course notes, textbook, course website resources, course videos. You may not use the internet beyond the course websites. You may not ask anyone else for help (except your professor), including other humans, or posting/entering your question into the internet. You may not share the questions or answers with anyone else.

Have you read, understood, and followed the honor code rules above?

YES / NO

Please write your best written proof of the following theorem. You will be graded on logic as well as writing.

Theorem 1. *Let a and b be integers. Suppose a^2 divides b and b^2 divides a. Then $a, b \in \{-1, 0, 1\}$.*

Hints: It's ok to use as a "known fact" that if $xy = 1$ for some integers x and y, then either $x = 1$ or $x = -1$.

Proof. Suppose that $a, b \in \mathbb{Z}$ and that $a^2 \mid b$ and $b^2 \mid a$. Then we have integers k and ℓ such that

$$a^2 k = b, \quad b^2 \ell = a.$$

First, we will show that $a \in \{0, 1, -1\}$. Combining these equations (plugging the first in the second),

$$a^4 k^2 \ell = a.$$

If $a = 0$, then $a \in \{0, -1, 1\}$, so we may assume $a \neq 0$. Then,

$$a^3 k^2 \ell = 1.$$

Since a and $a^2 k^2 \ell$ are integers that multiply to 1, $a \in \{1, -1\}$.

Since the problem is symmetrical in a and b, the same reasoning applies to show that $b \in \{0, 1, -1\}$. □

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If $a = 0$, then $a \in \{0, -1, 1\}$, so we may assume $a \neq 0$. This implies that $\ell \neq 0$ and $b \neq 0$ (using the equation $b^2 \ell = a$ above). Then, similarly, $k \neq 0$. So we can write

$$a^3 = 1/k^2 \ell.$$

Since $a \in \mathbb{Z}$, the fraction on the right hand side must evaluate to an integer. This means the denominator is ± 1, and we have that $a^3 \in \{1, -1\}$. But then $a \in \{1, -1\}$.

Since the problem is symmetrical in a and b, the same reasoning applies to show that $b \in \{0, 1, -1\}$. □