Math 2001, January 14, 2020

A very first proof

Definition. *An integer n is called* even *if it has the form $n = 2k$ for some integer k.*

Theorem. *If n is an even integer, then n^2 is an even integer.*

Proof. Suppose n is an even integer.
Then n has the form $n = 2k$ for some integer k.
Therefore, squaring both sides,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since k is an integer, so is $2k^2$.
Therefore n^2 has the form $n^2 = 2\ell$, where $\ell = 2k^2$ is an integer.
Therefore, n^2 is even. □

Some other phrasings

Definition 1. *Let n be an integer. We say that n is* even *if it is equal to twice some other integer.*

Theorem 2. *The square of an even integer is even.*

Proof. Assume that n is an even integer.
Then we may write $n = 2k$ with k being an integer.
Squaring both sides, we see that $n^2 = (2k)^2$.
We may rewrite this as $n^2 = 2(2k^2)$.
Now, we have expressed n^2 as twice the integer $2k^2$.
Therefore, n^2 is even. □

Proof. Let n be an even integer.
Then, by the definition of an even integer, $n = 2k$ for some $k \in \mathbb{Z}$.
Squaring, we obtain $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
Note that $2k^2$ is an integer, since it is a product of integers.
Therefore $n^2 = 2r$ with $r = 2k^2 \in \mathbb{Z}$.
Hence n^2 is even, by the definition of an even integer. □

The wordy proof

Let us prove the theorem together.

For the sake of argument, please imagine that you have selected an even integer, and call it n. It could be any even integer at all.

Now, since it is even, it must be twice another integer. Let's call the new one k. In other words, n is twice k, which we can write as $n = 2k$.

Next, the equation $n = 2k$ can be squared on both sides to yield another correct equation, namely: $n^2 = (2k)^2$.

We can rewrite the right-hand side of this somewhat, so that we get $n^2 = 2(2k^2)$.

So far, what we've learned is that for whatever even n you picked, there is an integer k so that together they satisfy this equation, namely, $n^2 = 2(2k^2)$.

But this equation says that n^2 is twice $2k^2$.

And $2k^2$ is an integer, since it is a product of integers.

So n^2 is twice an integer.

But that means n^2 is even.

So, let's sum up the argument. I have described a method whereby, no matter which even integer n you had in mind, I showed you how to see that n^2 is even.

More precisely, given that you knew what to double to obtain n, I was able to use that knowledge to show you what to double to obtain n^2. I gave you recipe for turning knowledge of n into knowledge of n^2: if $n = 2k$ then $n^2 = 2(2k^2)$.

An example run through

You pick 6. Six is even since it is twice 3. So we will apply the proof recipe our example, $n = 6$ and $k = 3$.

In particular, the proof says that n^2 should be twice $2k^2$. That is, 36 should be twice 18. It worked!