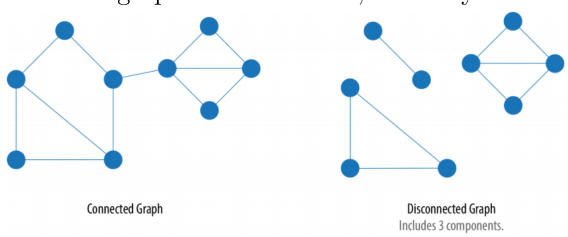
Math 2001, Spring 2023. Katherine E. Stange.

1 Assignment

Recall that a graph is *connected* if, for every two vertices, there is a path along edges from one to the other.



Prove the following theorem.

Theorem 1. *Let G be a graph with v vertices and e edges. If $e < v - 1$, then G is not connected.*

Upon closer investigation, we need to slightly modify the theorem.

Theorem 2. *Let G be a graph with $v > 1$ vertices and e edges. If $e < v - 1$, then G is not connected.*

Proof by induction. We will induct on the number of vertices. We begin with $v = 2$, in which case $e = 0$ and so we have two vertices and no edges, which is disconnected.

For the inductive step, assume that all graphs with fewer than N vertices satisfy the theorem. Let G be a graph with N vertices and $e < v - 1$.

First, we reduce to the case that G has some vertex of degree 1. Since $e < v - 1$, the average vertex degree is $2e/v < (2v - 2)/v < 2$. Hence there is some vertex v_0 with degree 0 or 1. If it has degree 0, then G is disconnected (since it has at least two vertices), in which case we are done. So we may assume v_0 has degree 1.

Removing this vertex v_0 and the single edge e_0 it touches, we obtain a smaller graph G' for which $e < v - 1$ still holds, and for which $v > 1$ (since G had at least 3 vertices).

By the inductive hypothesis, G' is not connected. In other words, there are two vertices w_0 and w_1 of G' for which there is no path between them. By the degree of v_0, observe that the edge e_0 cannot figure in any path between w_0 and w_1 in G. Hence G is also disconnected. □

There's a variation on induction called *minimal counterexample*. Here's the same proof in that language, in case you are interested (Hammack, Chapter 10.3).

Proof by minimal counterexample. We will prove this by contradiction. Assume there are counterexamples. Organize the counterexamples by size (number of vertices) and choose one whose size is the smallest in the list (there may be several of the same size; choose one). In other words, choose a connected graph G which is smallest (in terms of number of vertices) for which $e < v - 1$ and $v > 1$.

First, we show that G must have at least 3 vertices. For, if G has two vertices, then it has zero edges, and is therefore disconnected, a contradiction.

Next, we show that G must have some vertex of degree 1. Since $e < v - 1$, the average vertex degree is $2e/v < (2v - 2)/v < 2$. Hence there is some vertex v_0 with degree 0 or 1. If it has degree 0, then G is disconnected (since it has at least two vertices), a contradiction. So v_0 has degree 1.

Removing this vertex v_0 and the single edge e_0 it touches, we obtain a smaller graph G' for which $e < v - 1$ still holds, and for which $v > 1$ (since G had at least 3 vertices).

By the degree of v_0, observe that the edge e_0 cannot figure in any path between two vertices, unless one of the two vertices is v_0. Having removed e_0 and v_0, we have therefore not disconnected any two vertices. So this new smaller graph G' is also connected.

This contradicts the fact that the example we had was the minimal example. □