Math 2001, Spring 2023. Katherine E. Stange.

1 Assignment

Prove the following theorem.

Theorem 1. *Let f_n be the n-th Fibonacci number. That is, $f_1 = f_2 = 1$ and $f_{n+2} = f_{n-1} + f_n$ for $n \geq 1$. For all $n \geq 2$, we have $f_n < 2^n$.*

Proof. We will prove this by induction on n.

Base cases: Let $n = 2$. Then $f_2 = 1 < 2^2 = 4$. Let $n = 3$. Then $f_3 = f_2 + f_1 = 1 + 1 = 2 < 2^3 = 8$.

Inductive step: Suppose the theorem holds for $2 \leq n \leq k$, were $k \geq 3$. We will prove that it holds for $n = k+1$. Using the inductive hypothesis for $n = k$ and $n = k - 1$, we have

$$f_{k+1} = f_k + f_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}.$$

□

Questions for you: Why did I need two base cases?? Were you careful about which cases you were assuming in the inductive hypothesis?