MATHEMATICS 2001
WORKSHEET ON PIGEONHOLE PRINCIPLE

Prove it! For each theorem, please give an example illustrating the theorem, if possible, before proving it. For example, for Theorem 2, choose 3 random positive integers and verify that two have the same parity.

Theorem 1. *Let n and k be integers. Suppose n pigeons are to be placed into k holes. Suppose $n > k$. Then at least one hole is shared (i.e. has more than one pigeon in it).*

Theorem 2. *Among any 3 positive integers, there exist two of the same parity.*

Theorem 3. *Suppose* 5 *points are placed in the interior of a square with unit sides. Then some pair of these points is at distance less than or equal to* $1/\sqrt{2}$.

Theorem 4. *Amongst any n positive integers, there exist two whose difference is divisible by $n - 1$.*

Theorem 5. *Suppose n people are in a room together. Suppose each pair is either a pair of friends or not. Then there are two people with the same number of friends.*

Theorem 6. *Any $X \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|X| = 5$ will include two elements a and b such that $a + b = 9$.*

For this problem, define an *L-shaped region* of a chessboard to be a 5-square capital L shaped region, i.e. one corner square, plus two squares above it and two squares to the right.

Theorem 7. *No matter how one colours an* 8×8 *chessboard with black and white, there will always be two L-shaped regions that have the same colouring.*

Theorem 8. *Let* n *be a natural number. Then there exist distinct natural numbers* a *and* b *such that* $n^a - n^b$ *is divisible by* 10.

Theorem 9. *Given five distinct lattice points in the plane, at least one of the line segments defined as joining two such points has a lattice point as a midpoint.*

Card Problem. Consider this magic trick: A magician asks an audience member to pick five cards, which are not shown to the magician. The magician's accomplice looks at the cards, picks four of the cards, and shows these four to the magician in an order of his choosing. The magician then correctly guesses the fifth card.

Can you figure out a mathematical way to perform this trick? If so, I'll ask you to demonstrate it on your classmates! Hint: The pigeonhole principle guarantees that such a trick is possible. But it is challenging to come up with a good way to do it.

Hints!

(1) Hint: Use contradiction.
(2) Hint: The two pigeonholes are labelled “even” and “odd”. Put the three numbers into the two pigeonholes. What happens if two numbers are in the same pigeonhole?
(3) Hint: Yes, this is the same problem we saw sometime in past! Revisit it with pigeonhole. Break the square into four equal squares of side $1/2$, and place the five points (pigeons) into the four regions (pigeonholes).
(4) Hint: This is a generalization of Theorem 2. The pigeonholes are the possible remainders when divided by $n-1$, i.e. $0, 1, \ldots, n-2$. What if two integers end up in the same pigeonhole?
(5) Hint: Suppose “Pigeonhole 0” contains the people who are friendless, “Pigeonhole 1” contains those people having only one friend, “Pigeonhole 2” contains those people having exactly two friends, and so on. Place the n people (pigeons) into these n pigeonholes according to their “friendliness”. Is it really possible to have one person in each pigeonhole? Why not?
(6) Hint: First, try some examples. Then, think how to use pigeonhole.
(7) Card: you figure it
(8) Last digit of n^1, n^2 etc.
(9) midpoint is $((a+c)/2, (b+d)/2)$, consider parity of a, b, c, d.
(10) try colourings vs. places you could put an ℓ.