Take One or Two

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Consider the following game. The board consists of n tokens. At each turn, a player must take either 1 or 2 tokens off the board (which are discarded). The player who takes the last token(s), leaving 0 tokens, is the winner.

Definition 1. *A board with n tokens is called a* winning position *if the first player to move can always win the game, no matter how his opponent moves. Otherwise it is called a* losing position*.*

In other words, n is a winning position if the first player, provided he is smart enough to make all the right moves, can always win, no matter what the second player does. It is a losing position if the second player can, by a judicious choice of moves, win the game, no matter what the first player does.

Fact 1. *If the first player can leave the second player with a losing position, then the current position is winning.*

Fact 2. *If the first player has no choice but to leave the second player with a winning position, then the current position is losing.*

Theorem 1. *Let n be a positive integer. If n is divisible by 3, then n is a losing position. Otherwise n is a winning position.*

Proof. Suppose $n = 1$ or $n = 2$. Then the first player can take all the tokens and win immediately. Therefore n is a winning position.

Suppose $n = 3$. No matter how many tokens the first player takes, the second player is left with either 1 or 2 tokens, both of which are winning positions. Therefore n is a losing position.

Suppose $n = 4$ or $n = 5$. Then the first player can take all but 3 tokens, leaving his opponent with a losing position. Therefore n is a winning position.

Suppose $n = 6$. No matter how many tokens the first player takes, the second player is left with either 4 or 5 tokens, both of which are winning positions. Therefore n is a losing position.

Suppose $n = 7$ or $n = 8$. Then the first player can take all but 6 tokens, leaving his opponent with a losing position. Therefore n is a winning position.

Suppose $n = 9$. No matter how many tokens the first player takes, the second player is left with either 7 or 8 tokens, both of which are winning positions. Therefore n is a losing position.

Suppose $n = 10$ or $n = 11$. Then the first player can take all but 9 tokens, leaving his opponent with a losing position. Therefore n is a winning position.

Suppose $n = 12$. No matter how many tokens the first player takes, the second player is left with either 10 or 11 tokens, both of which are winning positions. Therefore n is a losing position.

Continue forever....

□

Proof. We prove this **by induction**.

Base Case: Suppose $n = 1$ or $n = 2$.

Then the first player can take all the tokens and win immediately. Therefore n is a winning position.

Suppose $n = 3$. No matter how many tokens the first player takes, the second player is left with either 1 or 2 tokens, both of which are winning positions. Therefore n is a losing position.

Inductive Step: Suppose that for all positive $k < n$, k is a winning position if and only if it is not divisible by 3 (this is the **inductive hypothesis**). We will show that n is a winning position if and only if it is not divisible by 3.

Suppose n is divisible by 3. No matter how many tokens the first player takes, the second player is left with a number that is not divisible by 3, which is a winning position **by the inductive hypothesis**. Therefore n is a losing position.

Suppose n is not divisible by 3. Then $n = 3\ell + k$ for some integers ℓ and k such that $\ell \geq 0$ and $k \in \{1, 2\}$. Then the first player can take k tokens, leaving his opponent with $3\ell < n$ tokens, which is a losing position **by the inductive hypothesis**. Therefore n is a winning position.

□

INDUCTIVE PROOF TEMPLATE

Theorem 2. *For all positive integers n, $P(n)$ is true.*

Proof. We will prove $P(n)$ is true by induction.

Base Case: Suppose $n = 1$. *Note: sometimes more.*

Insert a proof that $P(n)$ is true (under the assumption $n = 1$).

Inductive Step: Suppose that for all positive $k < n$, $P(k)$ is true.

Insert a proof that $P(n)$ is true (under the assumption that $P(k)$ is true for all $k < n$). □

UNROLLING AN INDUCTIVE PROOF

Theorem 3. *$P(4)$ is true.*

Proof. $P(1)$ is true by the Base Case.

$P(2)$ is true by the Inductive Step, since $P(1)$ is true.

$P(3)$ is true by the Inductive Step, since $P(2)$ and $P(1)$ are true.

$P(4)$ is true by the Inductive Step, since $P(3)$, $P(2)$ and $P(1)$ are true. □