Worksheet on Functions

March 10, 2020

1 Functions: terminology

- A function $f : A \to B$ is a way to assign one value of B to each value of A. A is the *domain*. B is the *codomain*.
- The function $f : A \to A$ that takes $f(a) = a$ for every $a \in A$ has a special name: *the identity function*.
- The *image* of a subset $X \subset A$ is the set of things X goes to in B, i.e.

$$f(X) = \{f(x) : x \in X\}.$$

- The image of A has a special name, the *range*.
- The *pre-image* of a subset $Y \subset B$ is the set of things that go into Y, i.e.

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$

- A function $f : A \to B$ is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is *onto*.)
- A function $f : A \to B$ is *injective* if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is 1-*to*-1.)
- A function $f : A \to B$ is *bijective* if it is both surjective and injective.

2 Practice problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
   (a) What is the domain? $\mathbb{R}$
   (b) What is the codomain? $\mathbb{R}$
   (c) What is the range? $[0, \infty)$
   (d) Is this the identity function? (Yes/No) no
   (e) Is this injective? no
   (f) Is this surjective? no
   (g) Is this bijective? no
   (h) What $f([0, 1])$? $[0,1]$
   (i) What is $f^{-1}([4, \infty))$? $(-\infty,-2] \cup [2,\infty) = \{x \in \mathbb{R} : |x| \geq 2\}$
2. Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(x) = 3x$.
   (a) What is the domain? $\mathbb{Z}$
   (b) What is the codomain? $\mathbb{R}$
   (c) What is the range? (use set builder) $\{3x : x \in \mathbb{Z}\}$
   (d) Is this the identity function? (Yes/No) No.

(e) Is this injective? yes

(f) Is this surjective? no

(g) Is this bijective? no

-2, -1, 0, 1, 2

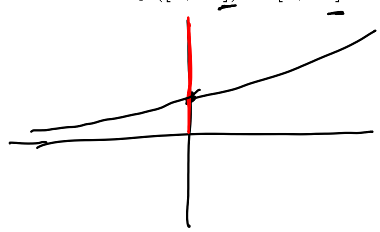
(h) What $f(\{x \in \mathbb{Z} : |x| < 3\})$? $\{0, 3, 6, -3, -6\}$

(i) What is $f^{-1}(5\mathbb{Z})$? Note that $5\mathbb{Z}$ is notation for $5\mathbb{Z} = \{5x : x \in \mathbb{Z}\}$. $5\mathbb{Z}$.

3. Draw a graph of a function which is injective but not surjective, which has domain and codomain $\mathbb{R}$, and satisfies $f([0, \infty]) = [1, \infty]$ and $f^{-1}([0, \infty]) = \mathbb{R}$.

y-axis

$f : \mathbb{R} \to \mathbb{R}$



4. Given two sets of equal cardinality $|A| = |B| = n$.

$n^n = \underbrace{n \cdots n}_{n \text{ times}} = \left(\begin{array}{c}\#\text{ of places to send 1st elt of } A\end{array}\right) \times \left(\begin{array}{c}\# \text{ of} \\ \vdots \\ \text{2nd}\end{array}\right) \times \cdots$

(a) How many functions are there $f : A \to B$?

(b) How many of these are bijective? $n!$

(c) Can you construct one which is injective but not bijective? Not possible.

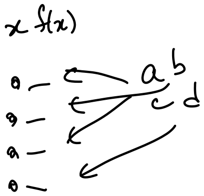
(not surjective)

$A = \{x, y\}$ $B = \{a, b\}$

x	a	b
y	b	a

5. Let A and B be finite sets, and suppose $f : A \to B$.

Fill in the table with P (possible) and I (impossible).



	$\lvert A\rvert = \lvert B\rvert$	$\lvert A\rvert > \lvert B\rvert$	$\lvert A\rvert < \lvert B\rvert$
bijective			
surjective, not injective			
injective, not surjective			
neither injective nor surjective			

6. Prove that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $2x + 3$ is injective, but not surjective.