Worksheet on Functions 2 – Solutions

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1 Functions: terminology

- A function $f : A \to B$ is a way to assign one value of B to each value of A. A is the *domain*. B is the *codomain*.
- The function $f : A \to A$ that takes $f(a) = a$ for every $a \in A$ has a special name: *the identity function*.
- The *image* of a subset $X \subset A$ is the set of things X goes to in B, i.e.
$$f(X) = \{f(x) : x \in X\}.$$
- The image of A has a special name, the *range*.
- The *pre-image* of a subset $Y \subset B$ is the set of things that go into Y, i.e.
$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$
- A function $f : A \to B$ is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is *onto*.)
- A function $f : A \to B$ is *injective* if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is 1-*to*-1.)
- A function $f : A \to B$ is *bijective* if it is both surjective and injective.

2 Practice problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
   (a) What is the domain? $\mathbb{R}$
   (b) What is the codomain? $\mathbb{R}$
   (c) What is the range? $[0, \infty]$
   (d) Is this the identity function? No.
   (e) Is this injective? No. (e.g. 2 and -2 go to 4)
   (f) Is this surjective? No. (e.g. nothing goes to -1)
   (g) Is this bijective? No.
   (h) What $f([0, 1])$? $[0, 1]$
   (i) What is $f^{-1}([4, \infty])$? $(-\infty, -2] \cup [2, \infty)$
2. Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(x) = 3x$.
   (a) What is the domain? $\mathbb{Z}$
   (b) What is the codomain? $\mathbb{R}$
   (c) What is the range? $\{3x : x \in \mathbb{Z}\}$.
   (d) Is this the identity function? No.

(e) Is this injective? Yes.

(f) Is this surjective? No. (e.g. $\sqrt{2}$, π or even 1 are not in the range.)

(g) Is this bijective? No.

(h) What $f(\{x \in \mathbb{Z} : |x| < 3\})$? $\{f(-2), f(-1), f(0), f(1), f(2)\} = \{-6, -3, 0, 3, 6\}$.

(i) What is $f^{-1}(5\mathbb{Z})$? Note that $5\mathbb{Z}$ is notation for $5\mathbb{Z} = \{5x : x \in \mathbb{Z}\}$. Answer: $\{x \in \mathbb{Z} : f(x) \text{ is a multiple of } 5\} = \{x \in \mathbb{Z} : 5 \mid 3x\} = \{x \in \mathbb{Z} : 5 \mid x\} = 5\mathbb{Z}$.

3. Draw a graph of a function which is injective but not surjective, which has domain and codomain $\mathbb{R}$, and satisfies $f([0,\infty]) = [1,\infty]$ and $f^{-1}([0,\infty]) = \mathbb{R}$.

4. Given two sets of equal cardinality $|A| = |B| = n$.

(a) How many functions are there $f : A \to B$? n^n since each of n items in A has n possible places to go.

(b) How many of these are bijective? To be bijective, you can't reuse an output. So essentially, we are ordering the elements of B in the second column of the table representing the function. That means, the number of bijective functions is $n!$.

(c) Can you construct one which is injective but not bijective? Equivalently, that would mean injective, but not surjective. Injective means you can't reuse the outputs while building a function table. But there are exactly n outputs, so unless you reuse some, so won't have any left over. So it will be surjective if it is injective. So: the answer is no.

5. Let A and B be finite sets, and suppose $f : A \to B$.

Fill in the table with P (possible) and I (impossible).

	$\|A\| = \|B\|$	$\|A\| > \|B\|$	$\|A\| < \|B\|$
bijective	P	I	I
surjective, not injective	I	P	I
injective, not surjective	I	I	P
neither injective nor surjective	P	P	P

6. Prove that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $2x + 3$ is injective, but not surjective.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = 2x + 3$.

We will show that f is injective. Suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned} 2x_1 + 3 &= 2x_2 + 3 \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

Hence $x_1 = x_2$ and therefore f is injective.

Now we will show that f is not surjective. Consider $y = 2$. This y is in the codomain. If $f(x) = y$, we would have

$$2x + 3 = 2$$

which implies $x = -1/2$. Since this value of x is not in the domain, there is no value of x in the domain with $f(x) = y$. Hence f is not surjective.