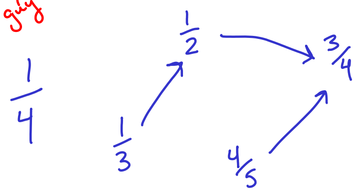
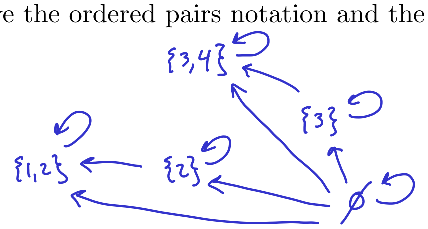
Example relations problems (Katherine E. Stange, Math 2001, CU Boulder)

don't forget this lonely guy

1. Draw an arrow diagram of the relation $R = \{(1/2, 3/4), (4/5, 3/4), (1/3, 1/2)\}$ on the set $A = \{1/2, 1/3, 1/4, 3/4, 4/5\}$.



2. Give the ordered pairs notation and the arrow diagram for the relation $\subseteq$ on $A = \{\emptyset, \{3,4\}, \{1,2\}, \{2\}, \{3\}\}$.



$\{(\emptyset, \{2\}), (\emptyset, \{3\}), (\emptyset, \{3,4\}), (\emptyset, \{1,2\}), (\{3\}, \{3,4\}), (\{2\}, \{1,2\}), (\emptyset, \emptyset), (\{2\}, \{2\}), (\{3\}, \{3\}), (\{1,2\}, \{1,2\}), (\{3,4\}, \{3,4\})\}$

3. Give an example of a relation on the set $A = \{e\}$. Give it as a set of ordered pairs, or as an arrow diagram (your choice).

$\{(e, e)\}$

4. Now give another, different, example of a relation on the set $A = \{e\}$.

e $\emptyset$

5. For each of the following relations, determine if it is reflexive, symmetric, transitive and/or equivalence.

   (a) The relation $\leq$ on $\mathbb{Z}$.
   - Reflexive? YES / NO
   - Symmetric? YES / NO
   - Transitive? YES / NO
   - Equivalence? YES / NO

   (b) The relation $\neq$ on $\{0\}$.
   - Reflexive? YES / NO
   - Symmetric? YES / NO
   - Transitive? YES / NO
   - Equivalence? YES / NO

   vacuously true (*)

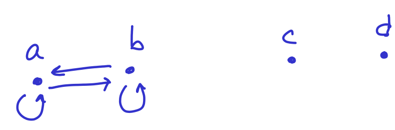
   arrow diagram: 0 (no arrows)

   (c) The relation $\{(e, e)\}$ on $\{e\}$.
   - Reflexive? YES / NO
   - Symmetric? YES / NO
   - Transitive? YES / NO
   - Equivalence? YES / NO

   arrow diag:

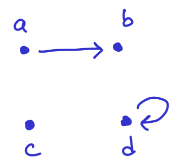
   (d) The relation $\{(a, a), (b, b), (a, b), (b, a)\}$ on $\{a, b, c, d\}$.
   - Reflexive? YES / NO
   - Symmetric? YES / NO
   - Transitive? YES / NO
   - Equivalence? YES / NO

   (c,c) missing



6. Give a relation on the set $\{a, b, c, d\}$ which is not reflexive and not symmetric. (You can give an arrow diagram or a set of ordered pairs.)

eg.

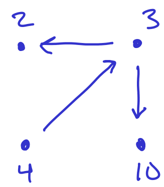


(there are many examples, but must include a) a dot w/ no self arrow  b) an arrow w/ no back arrow )

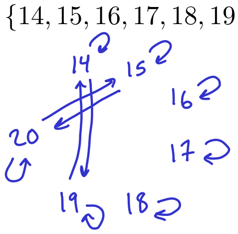
7. What is the equivalence class of $\{1\}$ under the equivalence relation "has the same cardinality as" on the set $\mathscr{P}(\{1, 2, 3\})$?

$\{ \{1\}, \{2\}, \{3\} \}$ (all subsets of $\{1,2,3\}$ of size one)

8. Draw an arrow diagram of the relation $R = \{(3, 10), (4, 3), (3, 2)\}$ on the set $A = \{2, 3, 4, 10\}$.



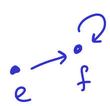
9. Give the ordered pairs notation and the arrow diagram for the relation $\equiv \pmod 5$ (in english, *equivalence modulo* 5) on $A = \{14, 15, 16, 17, 18, 19, 20\}$.



$\{ (14,14), (15,15), (16,16), (17,17), (18,18), (19,19), (20,20), (14,19), (19,14), (15,20), (20,15) \}$

10. Give an example of a relation on the set $A = \{e, f\}$. Give it as a set of ordered pairs ***and*** as an arrow diagram.

eg.



$\{ (e,f), (f,f) \}$ (there are many possible answers)

11. For each of the following relations, determine if it is reflexive, symmetric, transitive and/or equivalence.

(a) The relation $\geq$ on $\mathbb{R}$.

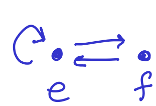
- Reflexive? YES / NO
- Symmetric? YES / NO
- Transitive? YES / NO
- Equivalence? YES / NO

(b) The relation $\neq$ on $\mathbb{Z}$.

- Reflexive? YES / NO $\quad 1 = 1$
- Symmetric? YES / NO
- Transitive? YES / NO $\quad 1 \neq 2,\ 2 \neq 1,\ 1 = 1$
- Equivalence? YES / NO

(c) The relation $\{(e, e), (f, e), (e, f)\}$ on $\{e, f\}$.

- Reflexive? YES / NO $\quad f$
- Symmetric? YES / NO
- Transitive? YES / NO $\quad (f,e), (e,f)$ but not (f,f)
- Equivalence? YES / NO



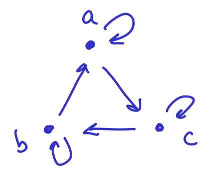
(d) The relation *sends* 0 *to the same place* on the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

eg. $f(x) = x^2 + 1$ and $f(x) = x + 1$ send 0 to 1

- Reflexive? YES / NO
- Symmetric? YES / NO
- Transitive? YES / NO
- Equivalence? YES / NO

12. Give a relation on the set $\{a, b, c\}$ which is reflexive and not symmetric. (You can give an arrow diagram or a set of ordered pairs; your choice.)

eg.



There are many correct solutions.
It must: a) have self arrows 
b) have some arrow without back arrow

13. What is the equivalence class of 2 under the equivalence relation *has the same last digit* on the set $\{x \in \mathbb{Z} : 0 \leq x \leq 50\}$?

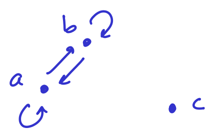
$\{2, 12, 22, 32, 42\}$ (everything ending in 2)

14. How many different equivalence classes are there in total, for the equivalence relation of the last question?

10 (one for each final digit $0, 1, 2, \ldots, 9$)

15. Give an example of an equivalence relation with equivalence classes whose cardinalities are not all the same. Give an example of two classes with different cardinalities.

ex 1:



$[a] = \{a, b\}$ size 2 $[c] = \{c\}$ size 1

ex 2: "$\equiv$ modulo 3" on $\{0, 1, 2, 3\}$

$[0] = \{0, 3\}$ size 2

$[1] = \{1\}$ size 1

16. How many different equivalence relations are possible on a set of 3 elements? Why?

How many ways to partition $\{a, b, c\}$? Partitions are the same as equivalence relations

1: $\{a, b, c\}$

2: $\{a\}, \{b, c\}$

3: $\{b\}, \{a, c\}$

4: $\{c\}, \{a, b\}$

5: $\{a\}, \{b\}, \{c\}$.

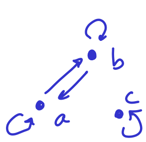
17. Suppose A has cardinality n, and suppose there is an equivalence relation on A whose m equivalence classes all have the same cardinality. What size are the equivalence classes?

$$\frac{n}{m} = \frac{\text{total \# of elements}}{\text{\# of classes}} = \text{size of each class}$$

18. If R and S are two equivalence relations on a set A, is $R \cup S$ an equivalence relation on A? Why or why not?

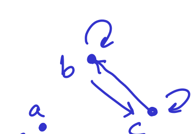
NO!

Counter-example:



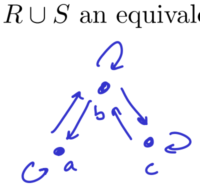
R

$\{(a,a), (b,b), (c,c), (a,b), (b,a)\}$



S

$\{(a,a), (b,b), (c,c), (b,c), (c,b)\}$



$R \cup S$

$\{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

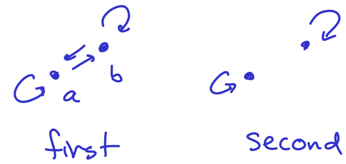
not transitive.

19. How many different partitions are there of a two-element set?

two: $\{a, b\}$ or $\{a\}, \{b\}$

20. How many different equivalence relations are there on a two-element set?

two, because of previous problem:



21. Give an example of a relation on $\mathbb{Z}$ which is not transitive.

$\neq$ since $1 \neq 2$ and $2 \neq 1$ but $1 = 1$