Example logic problems (Katherine E. Stange, Math 2001, CU Boulder)

1. Give the truth table for the expression $P \implies Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

I prefer the rows in this order, but it isn't strictly necessary.

2. Give the truth table for the expression $(P \vee Q) \implies (\sim R)$. For partial credit, show your intermediate columns (e.g. $\sim R$). Note: this has 3 variables, so it should have 8 rows.

P	Q	R	$P \vee Q$	$\sim R$	$(P\vee Q) \Rightarrow (\sim R)$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

extra horizontal lines just to keep it straight (not necessary)

intermediate columns

3. Give an example of a boolean expression in two variables P and Q which is a *tautology*, in other words, in its truth table, it evaluates to T (true) for all inputs. Note: the expression must involve both variables somehow.

e.g. $P \vee (\sim P) \vee Q$ or $(P \Leftrightarrow P) \vee Q$ etc.

or $(P \Leftrightarrow Q) \vee (P \Leftrightarrow \sim Q)$

many possible correct answers

4. Give an example of a boolean expression in two variables P and Q which is a *contradiction*, in other words, in its truth table, it evaluates to F (false) for all inputs. Note: the expression must involve both variables somehow.

e.g. $P \wedge (\sim P) \wedge Q$ etc.

or $(P \Leftrightarrow Q) \wedge (P \Leftrightarrow \sim Q)$

many possible correct answers

5. For each statement, give the converse and the contrapositive. It is important that the sentences you write make english grammatical sense (e.g. don't use a variable before introducing it).

   (a) If $x \in \mathbb{Z}$ is prime, then x is odd.

   - Converse: If $x \in \mathbb{Z}$ is odd, then x is prime. ✓

     (Not so great: If x is odd, then $x \in \mathbb{Z}$ is prime. ✗)
   - Contrapositive: If $x \in \mathbb{Z}$ is even, then x is not prime.

(b) If $X \subseteq S$, then $|X| = 3$ or $|X| = 2$.

- Converse: If $|X|=3$ or $|X|=2$ then $X \subseteq S$.
- Contrapositive: If $|X| \neq 3$ and $|X| \neq 2$, then $X \nsubseteq S$.

(c) If the graph G has n vertices, then it has at least one edge.

- Converse: If the graph G has at least one edge, then it has n vertices.
- Contrapositive: If the graph G has no edges, then the number of vertices it has is not equal to n.

6. For each symbolic statement, determine if it is true or false.

(a) $\forall x \in \mathbb{Z}, x > 3$. TRUE, FALSE — there are integers less than or equal to 3

(b) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x = -y$. TRUE, FALSE — every integer has a negative

(c) $\exists x \in \mathbb{Q}, x^2 = 2$. TRUE, FALSE — $\sqrt{2}$ is irrational

(d) $\exists X \subseteq \mathbb{R}, 3 \in X$. TRUE, FALSE — there is a subset of the reals containing 3.

7. For each english statement, give the same statement in symbolic notation (**no english words**), including such things as quantifiers ($\exists$, $\forall$), boolean operators ($\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$), etc.

(a) The number a is an even integer.

$$\exists\, k \in \mathbb{Z},\ a = 2k.$$

(b) If $x \in \mathbb{Z}$, then x is even.

$$\underbrace{\forall x \in \mathbb{Z}}_{\text{for all } x},\ \underbrace{\exists k \in \mathbb{Z},\ x = 2k}_{x \text{ is even}}.$$

(c) Every subset of S has cardinality less than 6.

$$\forall\, X \subseteq S,\ |X| < 6.$$

(d) All rational numbers have rational squares.

$$\underbrace{\forall x \in \mathbb{Q}}_{\text{for all rational } x},\ \underbrace{x^2 \in \mathbb{Q}}_{\text{the square } x^2 \text{ is rational}}.$$

8. Negate each of the following statements. Simplify your negations to a reasonable level for full credit (that is, don't just give "It is false that X" as the negation of X; this is correct but not enlightening; try to "move the negation in" as far as you can, at the very least inside the quantifier.)

   (a) There are no polynomials of degree 7.

   There are polynomials of degree 7.

   (b) All polynomials of degree 6 have exactly 6 roots in the complex numbers.

   $\sim(\forall x, P(x))$
   $= \exists x, \sim P(x)$.

   There are polynomials of degree 6 which do not have exactly 6 roots in the complex numbers.

   (c) Among polynomials f with integer coefficients, if f has degree 3, then f^2 has degree 6.

   $\sim(P \Rightarrow Q)$
   $= P \wedge (\sim Q)$

   Among polynomials f with integer coefficients, there is an f with degree 3 but for which f^2 does not have degree 6.

   Or, more simply: there's a polynomial f with integer coefficients and degree 3, for which f^2 does not have degree 6.

   (d) There are polynomials having degree 3 and no rational roots.

   $\sim(\exists x, P(x))$
   $= \forall x, \sim P(x)$

   There are no polynomials having deg 3 and no rat'l roots.
   OR
   For all polynomials f, either f does not have degree 3 or f has a rational root.

   (e) Every polynomial has at least one root.

   Some polynomial has no roots.

   (f) If a polynomial has a root, then it has degree at least 1.

   There's a polynomial with a root but degree zero.

   there's no such thing as negative degree

9. True or False

   (a) $P \Rightarrow Q = P \vee (\sim Q)$ Nope, it's typically $P \Rightarrow Q = (\sim P) \vee Q$

   (b) $\sim(P \vee Q) = (\sim P) \vee (\sim Q)$ Nope, but $\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$

   (c) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$ Yes.

10. Fill in the missing symbols in the blanks (choose from $\Leftrightarrow$, $\Rightarrow$, $\wedge$ and $\vee$):

   (a) $\sim(P \Rightarrow Q) = P \underline{\ \wedge\ } (\sim Q)$

   (b) $\sim(P \wedge Q) = (\sim P) \underline{\ \vee\ } (\sim Q)$

   (c) $P \wedge (Q \vee R) = (P \underline{\ \wedge\ } Q) \underline{\ \vee\ } (P \underline{\ \wedge\ } R)$

   (d) $P \Leftrightarrow Q = (P \Rightarrow Q) \underline{\ \wedge\ } (Q \Rightarrow P)$

11. Draw lines connecting expressions in the first column to their negations in the second column. Everything has a pair.

$P \Rightarrow Q$	$(\sim P) \wedge (\sim Q)$
$(\sim P) \Rightarrow Q$	$P \wedge (\sim Q)$
$\sim(P \Leftrightarrow Q)$	$(\sim P) \vee (\sim Q)$
$P \wedge Q$	$(\sim P) \Leftrightarrow (\sim Q)$
$P \wedge (\sim Q)$	$(\sim P) \vee Q$

$= \quad P \vee Q$ De Morgan

$= \quad P \Rightarrow Q$ only false when $P = T$, $Q = F$

$= \quad P \Leftrightarrow Q$ both say P and Q agree

$= \quad Q \Rightarrow P$ contrapositive