Example functions problems (Katherine E. Stange, Math 2001, CU Boulder)

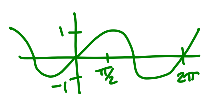
1. For each of the following functions, give the various items requested. You may need to use set builder notation.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2(x + 1)$.
   - Domain: $\mathbb{Z}$
   - Codomain: $\mathbb{Z}$
   - Range: $2\mathbb{Z}$ (even integers)
   - $f(\{-10, 10\})$: $\{2(-10+1), 2(10+1)\} = \{-18, 22\}$
   - $f^{-1}(\{4, 5, 6\})$: $\{1, 2\}$ since $f(1)=4$, $f(2)=6$

   (b) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$.
   - Domain: $\mathbb{R}$
   - Codomain: $\mathbb{R}$
   - Range: $(0, \infty)$
   - $g([0, 2])$: $[e^0, e^2] = [1, e^2]$
   - $g^{-1}([e, e^2])$: $[1, 2]$

   (c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x^2 - 1$.
   - Domain: $\mathbb{Z}$
   - Codomain: $\mathbb{Z}$
   - Range: $\{3x^2 - 1 : x \in \mathbb{Z}\} = \{-1, 2, 11, \ldots\}$
   - $f(\{1, 2\})$: $\{2, 11\}$
   - $f^{-1}(\{3, 4, 5\})$: $\emptyset$ (no $x \in \mathbb{Z}$ gives $3x^2 - 1 = 3, 4$ or 5)

   (d) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$.
   - Domain: $\mathbb{R}$
   - Codomain: $\mathbb{R}$
   - Range: $[-1, 1]$
   - $g([0, \pi/2])$: $[0, 1]$
   - $g^{-1}(\{\pi\})$: $\emptyset$ (no x has $\sin(x) = \pi$ because $\pi \notin$ range)



2. For each of the following functions, determine if it is injective, surjective or bijective.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 7x$.
   - Injective? YES / NO
   - Surjective? YES / NO $1 \neq 7x$ for any x
   - Bijective? YES / NO

   (b) $g : \mathbb{R} \to [-1, 1]$ given by $f(x) = \sin(x)$.
   - Injective? YES / NO
   - Surjective? YES / NO codomain $= [-1, 1] =$ range
   - Bijective? YES / NO

   (c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f((x, y)) = x + y$.
   - Injective? YES / NO $f((1,1)) = f((2,0)) = 2$
   - Surjective? YES / NO $f((x,0)) = x$ for any $x \in \mathbb{Z}$
   - Bijective? YES / NO

   (d) Let $X = \{a, b, c\}$. Let $k : \mathscr{P}(X) \to \mathscr{P}(X)$ given by $k(Y) = X - Y$ (that's a 'set minus' symbol).
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

   $\emptyset \longrightarrow \{a, b, c\}$
   $\{a\} \longrightarrow \{b, c\}$
   etc.

   (e) f: $\{a, b, c\} \to \{w, x, y, z\}$ given by $f(a) = w$, $f(b) = z$ and $f(c) = y$.
   - Injective? YES / NO
   - Surjective? YES / NO misses x
   - Bijective? YES / NO

(f) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x - 1$.

- Injective? YES / NO
- Surjective? YES / NO doesn't hit 0 for example
- Bijective? YES / NO

(g) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(h) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $f((x, y)) = (y, x)$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(i) Let $X = \{a, b, c\}$. Let $k : \mathscr{P}(X) \to \mathscr{P}(X)$ given by $k(Y) = Y \cup \{a\}$.

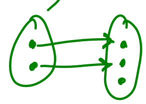
- Injective? YES / NO $k(\{a\}) = k(\emptyset) = \{a\}$.
- Surjective? YES / NO only hits sets containing a
- Bijective? YES / NO

(j) $f : \{a, b, c\} \to \{y, z\}$ given by $f(a) = y$, $f(b) = z$ and $f(c) = y$.

repeats

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

3. If a function f is injective, does that imply f is bijective? Why/why not?

(If $f: A \to B$ where $|A| \underset{\text{finite}}{=} |B|$, then yes.)
But in general not. For example  injective but not surjective

4. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

$f(x) = \lfloor \frac{x}{2} \rfloor = \begin{cases} x/2 & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$

floor function

then $f(2) = f(3) = 1$ (not injective)

but $f(2x) = x$ for all x (surjective)

5. Let A be a finite set. If a function $f : A \to A$ is injective, does that imply f is bijective?

Yes, since domain & codomain are finite, same size.

6. Give an example of a function $f : \mathbb{Z} \to \mathbb{R}$ which is injective but not surjective.

$f(x) = x$ If $f(x) = f(y)$ then $x = y$, so injective.
But no $x \in \mathbb{Z}$ has $f(x) = \pi$, so not surjective.

7. If a function $f : \mathbb{Z} \to \mathbb{Z}$ is surjective, does that imply f is bijective? Why/why not?

No. See example above (#4).
The set $\mathbb{Z}$ is not finite.

8. In the following problems, provide the composition or inverse requested if it exists; if not, then state "not defined". **Pay attention to domains and codomains!**

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$ and $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x + 3$.
   - $g \circ f$: technically not defined ($\mathbb{R} \neq \mathbb{Z}$)
   - $f \circ g$: $f \circ g(x) = |x+3|$
   - f^{-1}: not defined (f is not injective)
   - g^{-1}: not defined (g is not surjective)

   (b) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by given by $f((x, y)) = x + y$ and $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = x^2$.
   - $g \circ f$: $g \circ f((x,y)) = (x+y)^2$
   - $f \circ g$: not defined ($\mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z}$)
   - f^{-1}: not defined (f not injective)
   - g^{-1}: not defined (g not injective)

   (c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = 3x$ and $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = x + 10$.
   - $g \circ f$: $g \circ f(x) = 3x + 10$
   - $f \circ g$: $f \circ g(x) = 3(x+10)$
   - f^{-1}: not defined (f not surjective)
   - g^{-1}: $g^{-1}(x) = x - 10$.

   (d) Let $X = \{1, 2, 3\}$. Let $f : \mathscr{P}(X) \to \mathbb{Z}$ be given by $f(A) = |A|$ and $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x^2$.
   - $g \circ f$: $g \circ f(A) = |A|^2$
   - $f \circ g$: not defined ($\mathbb{R} \neq \mathscr{P}(X)$)
   - f^{-1}: not defined (f not injective or surjective)
   - g^{-1}: not defined (g not surjective or injective)

9. For the following functions, determine which compositions are defined, and for each composition that is defined, determine the composition.

   $f : \{1, 2, 3\} \to \{a, b\}$

x	$f(x)$
1	a
2	b
3	b

x	$g \circ f(x)$
1	1
2	2
3	2

   $g : \{a, b\} \to \{1, 2\}$

x	$g(x)$
a	1
b	2

x	$f \circ h(x)$
a	a
b	a
c	b

   $h : \{a, b, c\} \to \{1, 2, 3\}$

x	$h(x)$
a	1
b	1
c	3

10. For each of the following functions, determine if it has an inverse and what the inverse should be. If it has no inverse, explain why. If you give an inverse, prove it (meaning, compute the composition in both orders and check it is the identity).

    (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$. $f^{-1} : \mathbb{R} \to \mathbb{R}$, $f^{-1}(y) = \sqrt[3]{y-1}$

    $f^{-1} \circ f(x) = \sqrt[3]{(x^3+1)-1} = x$

    $f \circ f^{-1}(y) = (\sqrt[3]{y-1})^3 + 1 = y$

    (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$. no inverse (f is not injective or surjective)

    (c) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

x	$g(x)$
a	1
b	1
c	3

    no inverse (g is not injective or surjective)

    (d) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

x	$g(x)$
a	1
b	3
c	2

x	$g^{-1}(x)$
1	a
2	c
3	b

11. Let A and B be finite sets, and suppose $f : A \to B$.

Fill in the table with P (possible) and I (impossible).

	$\|A\| = \|B\|$	$\|A\| > \|B\|$	$\|A\| < \|B\|$
bijective	P	I	I
surjective, not injective	I	P	I
injective, not surjective	I	I	P
neither injective nor surjective	P	P	P

12. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = x^6$.

$g(x) = x^2$

13. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x + 1$. Find a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f(x) = x + 3$.

$g(x) = x + 2$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = 7x^3$.

$g(x) = 7x$

15. Suppose $f : A \to B$ is bijective. Does f have an inverse?

Yes. A function is bijective if and only if it has an inverse.

16. State the pigeonhole principle.

Version 1: Let $f: A \to B$ be a function. Suppose A, B are finite with $|A| > |B|$. Then f is not injective.

Version 2: Suppose there are n pigeons and k holes. If all pigeons are placed in holes and $n > k$, then some hole contains more than one pigeon.

17. State the definition of injectivity.

Let $f: A \to B$. Then f is injective if for all $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

⟨Note: contrapositive is ok too.⟩

18. State the definition of surjectivity.

Let $f: A \to B$. Then f is surjective if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$.

19. Given two sets of equal cardinality $|A| = |B| = n$.

(a) How many functions are there $f : A \to B$? n^n Each domain element has n choices. There are n domain elements.

(b) How many of these are bijective? $n!$ A bijection orders the elements of B to match those in A.

(c) Can you construct one which is injective but not bijective?

No. Since $|A| = |B|$ and the sets are finite, injectivity or surjectivity implies bijectivity.