# Example functions problems (Katherine E. Stange, Math 2001, CU Boulder)

1. For each of the following functions, give the various items requested. You may need to use set builder notation.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2(x+1)$.
   - Domain:
   - Codomain:
   - Range:
   - $f(\{-10, 10\})$:
   - $f^{-1}(\{4, 5, 6\})$:

   (b) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$.
   - Domain:
   - Codomain:
   - Range:
   - $g([0, 2])$:
   - $g^{-1}([e, e^2])$:

   (c) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x^2 - 1$.
   - Domain:
   - Codomain:
   - Range:
   - $f(\{1, 2\})$:
   - $f^{-1}(\{3, 4, 5\})$:

   (d) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$.
   - Domain:
   - Codomain:
   - Range:
   - $g([0, \pi/2])$:
   - $g^{-1}(\{\pi\})$:

2. For each of the following functions, determine if it is injective, surjective or bijective.

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 7x$.
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

   (b) $g : \mathbb{R} \to [-1, 1]$ given by $f(x) = \sin(x)$.
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

   (c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f((x, y)) = x + y$.
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

   (d) Let $X = \{a, b, c\}$. Let $k : \mathscr{P}(X) \to \mathscr{P}(X)$ given by $k(Y) = X - Y$ (that's a 'set minus' symbol).
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

   (e) $\ell : \{a, b, c\} \to \{w, x, y, z\}$ given by $f(a) = w$, $f(b) = z$ and $f(c) = y$.
   - Injective? YES / NO
   - Surjective? YES / NO
   - Bijective? YES / NO

(f) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 3x - 1$.
- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(g) $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$.
- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(h) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $f((x, y)) = (y, x)$.
- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(i) Let $X = \{a, b, c\}$. Let $k : \mathscr{P}(X) \to \mathscr{P}(X)$ given by $k(Y) = Y \cup \{a\}$.
- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(j) $\ell : \{a, b, c\} \to \{y, z\}$ given by $f(a) = y$, $f(b) = z$ and $f(c) = y$.
- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

3. If a function $f$ is injective, does that imply $f$ is bijective? Why/why not?

4. Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

5. Let $A$ be a finite set. If a function $f : A \to A$ is injective, does that imply $f$ is bijective?

6. Give an example of a function $f : \mathbb{Z} \to \mathbb{R}$ which is injective but not surjective.

7. If a function $f : \mathbb{Z} \to \mathbb{Z}$ is surjective, does that imply $f$ is bijective? Why/why not?

8. In the following problems, provide the composition or inverse requested if it exists; if not, then state "not defined". **Pay attention to domains and codomains!**

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$ and $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x + 3$.
   - $g \circ f$:
   - $f \circ g$:
   - $f^{-1}$:
   - $g^{-1}$:

   (b) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by given by $f((x, y)) = x + y$ and $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = x^2$.
   - $g \circ f$:
   - $f \circ g$:
   - $f^{-1}$:
   - $g^{-1}$:

   (c) Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = 3x$ and $g : \mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = x + 10$.
   - $g \circ f$:
   - $f \circ g$:
   - $f^{-1}$:
   - $g^{-1}$:

   (d) Let $X = \{1, 2, 3\}$. Let $f : \mathscr{P}(X) \to \mathbb{Z}$ be given by $f(A) = |A|$ and $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x^2$.
   - $g \circ f$:
   - $f \circ g$:
   - $f^{-1}$:
   - $g^{-1}$:

9. For the following functions, determine which compositions are defined, and for each composition that is defined, determine the composition.

   $f : \{1, 2, 3\} \to \{a, b\}$

   | $x$ | $f(x)$ |
   |---|---|
   | 1 | a |
   | 2 | b |
   | 3 | b |

   $g : \{a, b\} \to \{1, 2\}$

   | $x$ | $g(x)$ |
   |---|---|
   | a | 1 |
   | b | 2 |

   $h : \{a, b, c\} \to \{1, 2, 3\}$

   | $x$ | $h(x)$ |
   |---|---|
   | a | 1 |
   | b | 1 |
   | c | 3 |

10. For each of the following functions, determine if it has an inverse and what the inverse should be. If it has no inverse, explain why. If you give an inverse, prove it (meaning, compute the composition in both orders and check it is the identity).

    (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$.

    (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$.

    (c) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

    | $x$ | $g(x)$ |
    |---|---|
    | a | 1 |
    | b | 1 |
    | c | 3 |

    (d) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

    | $x$ | $g(x)$ |
    |---|---|
    | a | 1 |
    | b | 3 |
    | c | 2 |

11. Let $A$ and $B$ be finite sets, and suppose $f : A \to B$.

Fill in the table with $P$ (possible) and $I$ (impossible).

| | $\lvert A\rvert = \lvert B\rvert$ | $\lvert A\rvert > \lvert B\rvert$ | $\lvert A\rvert < \lvert B\rvert$ |
|---|---|---|---|
| bijective | | | |
| surjective, not injective | | | |
| injective, not surjective | | | |
| neither injective nor surjective | | | |

12. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = x^6$.

13. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x + 1$. Find a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f(x) = x + 3$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$. Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \circ f(x) = 7x^3$.

15. Suppose $f : A \to B$ is bijective. Does $f$ have an inverse?

16. State the pigeonhole principle.

17. State the definition of injectivity.

18. State the definition of surjectivity.

19. Given two sets of equal cardinality $|A| = |B| = n$.

    (a) How many functions are there $f : A \to B$?

    (b) How many of these are bijective?

    (c) Can you construct one which is injective but not bijective?