Example counting problems (Katherine E. Stange, Math 2001, CU Boulder)

1. Suppose your turn in a game depends on rolling one six-sided die and flipping one two-sided coin. How many outcomes are possible?

   Solution: First, roll the die (6 outcomes), then flip the coin (2 outcomes). The answer is

   $$6 \cdot 2.$$

2. There are six different vegetables on the counter. How many ways can you select one to use for a main course, and one to use for a dessert?

   Solution: The main course vegetable can be chosen from 6 options. Once that vegetable is removed for the main course, there are 5 ways to choose one for dessert. The answer is

   $$6 \cdot 5.$$

3. A doctor must visit 4 different patients A, B, C and D, in some order. In addition, for each patient, she must order or not order a test. How many ways can this happen? In other words, how many ways can she visit the patients in some order and decide on tests for each. (For example, one outcome is (1st visit C, test; 2nd visit B, no test; 3rd visit D, no test; 4th visit A, test).)

   Solution: There are 4! orders in which to visit the patients. For each ordering, there are four binary choices (test or no test) to make. The answer is

   $$4!2^4.$$

4. You are required to choose a password for a virtual hamster racing website (we all need something to keep us busy in the age of coronavirus, and we can't do it in reality anymore). The password must be 8 characters long. Seven of the characters are letters (26 letters to choose from), and one character is a 'special character' (meaning chosen from the 10 shown above the numbers on your keyboard). The special character can go anywhere among the others.

   Solution: First, choose the position for the special character. There are 8 positions. Next, choose the special character. There are 10 choices. Next, for each of the remaining empty spaces in order, choose one of 26 letters. The answer is

   $$8 \cdot 10 \cdot 26^7.$$

5. Consider a combination lock which uses the digits 0 through 9, where a combination involves exactly 4 digits, and the lock has the restriction that no two *consecutive* digits in a valid combination can be the same. (For example, 3346 is not allowed but 3463 is ok.) How many valid combinations are there?

   Solution: There are ten distinct digits. The first digit of the combination can be anything. The next digit can be chosen from 9 possibilities, since there is one possibility ruled out by the econsecutive rule. The same is true for each digit following. The answer is

   $$10 \cdot 9 \cdot 9 \cdot 9.$$

6. How many ways can you break a group of 13 racing hamsters up into *three* disjoint sets (empty sets are ok), called *Awesome Team*, *Mediocre Team* and *Tasty Team*? In other words, how many ways can you partition a set of 13 things into 3 disjoint differently-named subsets whose union is the full set?

   Solution: Each hamster must be placed on exactly one of the three teams. Working hamster-by-hamster, the solution is

   $$3^{13}.$$

7. How many ways can you make a 7-letter word by rearranging the letters IEEPVVV?

   Solution: There are 7! rearrangements of the letters, considering the repeats as distinct (e.g., labelling the V's as V_1, V_2 and V_3). However, this overcounts, since the repeated letters are not distinct. There are two E's, so we divide by 2!. There are 3 V's, so we also divide by 3!. We get

   $$\frac{7!}{3!2!}.$$

8. How many ways can you choose a subset of $\{a, b, c, d, e, f, g\}$, where the subset must have size 3 or 5, and must contain the element f?

   Solution: Since the subset must contain f, we can put that into the subset first and the problem becomes one of choosing a subset of $\{a, b, c, d, e, g\}$ of size 2 or 4. The set we are choosing from has size 6. The answer is

$$\binom{6}{2} + \binom{6}{4}.$$

9. You must pair up everyone in the class for a partner activity. There are 8 students. How many ways can the pairs be formed? Order doesn't matter (within pairs or among pairs).

   Solution: We can put everyone in the class in order in 8! ways. Having done that, we can pair the first two together, next two together, and so on. There will be a total of four pairs. Since the order doesn't matter inside each of the four pairs (i.e. Ted,Eve is the same as Eve, Ted), we divide by 2 four times. Since the order of the four pairs doesn't matter, we divide by 4!. The result is

$$\frac{8!}{4!2^4}$$

10. You roll two dice. How many different outcomes are there? Hint: The dice are identical, so the results are not ordered, i.e. (5,6) is the same as (6,5), but don't forget about doubles like (5,5)!

    Solution: Roll the first die (6 possible outcomes), then roll the second die (6 possible outcomes). Therefore there are 6! possible results, e.g. (3,2), (2, 3), (3,3) etc. However, for *those which are not doubles*, we will see them twice in the list (e.g. (3,2) and (2,3) will both appear), which is overcounting. There are six doubles, so we have

$$(\#\text{ doubles}) + (\#\text{ non-doubles})/2 = 6 + \frac{6^2 - 6}{2}.$$

11. How many n-digit binary sequences (sequences of 0s and 1s) contain exactly k 1s, but do not have a 1 in the first position?

    Solution: Since the first position is a 0, we can set that first and the problem becomes to find how many $n-1$-digit binary sequences have exactly k 1's. This means choosing exactly k positions to put a 1 (filling the other positions with zeroes), so the answer is

$$\binom{n-1}{k}.$$