Math 2001, Spring 2023. Katherine E. Stange.

1 Assignment

Suppose you have 7 cups, face down on a table. At each move, you may turn over exactly 4 cups (if the cup is up, you turn it down; if the cup is down, you turn it up). Prove that it is impossible to achieve a state where all 7 cups are face up with any finite number of such moves.

Hint: Think about the possible states you can achieve, in particular, their parity (even or oddness). Please try to write the most beautiful, simple, clear proof that you can.

2 A possible solution

At the beginning of the problem, the number of face-up cups is even. We will show that whenever four cups are turned over, the parity of the number of face-up cups is preserved. Since the goal is seven face-up cups (an odd number), it is impossible to achieve the goal.

It remains to prove that the parity is preserved. Suppose there are X face-up cups. When turning over four cups, we turn n cups from down to up and m cups from up to down, where $n + m = 4$. Therefore the number of face-up cups afterward is $X + n - m$. We must show that $n - m$ is even, which will imply that the parity of X is unchanged.

Since $n + m = 4$, the parity of n and m agree (either both odd, or both even). If they are both even, then $n - m$ is even. If they are both odd, then $n - m$ is even. Thus we have shown that $n - m$ is even, completing the proof.

Comment: this proof proceeds by simplifying the goal. Each paragraph moves the goal-post to a simpler task, until we complete it.

3 Another possible solution

Credit to David for this solution!

Consider the problem cup-by-cup. Each cub begins face-down and must end face-up. Therefore each cup must be turned over an odd number of times. The total number of cup-turns in a complete solution is therefore a sum of seven odd numbers. Therefore to solve the puzzle, the number of cup-turns in total must be odd.

However, each turn consists of an even number of cup-turns. Therefore the puzzle has no solution.

4 ChatGPT

I made ChatGPT a separate PDF just for your curiosity.