Counting Quiz (Katherine E. Stange, Math 2001, Spring 2023, CU Boulder)

Name:

For each counting problem, please determine the answer but preferably leave it unsimplified form. You may use binomial coefficients and factorials in your final answer if appropriate. Provide one or two sentences or phrases/point-form of brief justification.

If the questions are unclear, please ask during the test and I will clarify.

1. You are in charge of a restaurant and must set the menu for the week. Each weekday you must serve either tofu or pineapple for the main course, and either chocolate, sardines or okra for dessert. How many ways can you set the menu for the week (7 days)?

   Soln. Each weekday, there are two choices for the main course and three choices for the dessert, hence $2 \cdot 3$ choices for the menu. Since there are seven days, we do this seven times:

   $$2^7 \cdot 3^7.$$

2. You have 5 **different** hamsters. You must choose a subset of hamsters to give to your mother. You don't want her to be sad, so you must give her at least one of the hamsters. How many ways can this be done?

   Soln. There are 2^5 subsets of the hamsters, but you don't want to give her the empty set, so the answer is

   $$2^5 - 1.$$

3. You must lay out 7 **identical** cups on the table in a row. Exactly three of them must be right-side-up and the rest must be upside-down. How many different ways can this be done? (Examples: one solution is Up-Up-Down-Down-Down-Up-Down. But Up-Up-Up-Down-Down-Down-Up is not a valid solution because too many of them are up.)

   Soln. You must choose three out of 7 positions to turn up. (Or, equivalently, four positions to turn down.)

   $$\binom{7}{3} = \binom{7}{4}.$$

4. How many ways can you arrange 9 pencils in a row, where 3 are identical blue pencils, 3 are identical green pencils, and 3 are identical red pencils?

   Soln. You can arrange the pencils in 9! ways. However, if reorder the blue pencils, we can't tell; similarly for the green and red. Hence we are overcounting by a factor of 3!3!3!.

   $$\frac{9!}{3!3!3!}.$$

   An alternate solution is to imagine 9 spots on the table. Choose 3 to fill with blue pencils, which can be done in $\binom{9}{3}$ ways. Then choose 3 from the remaining 6 empty spots to fill with green pencils, which can be done in $\binom{6}{3}$ ways. Then fill the rest with red pencils. So we get

   $$\binom{9}{3}\binom{6}{3}.$$

   This is actually the same answer! Check it out:

   $$\binom{9}{3}\binom{6}{3} = \frac{9!}{3!6!}\frac{6!}{3!3!} = \frac{9!6!}{3!6!3!3!} = \frac{9!}{3!3!3!}.$$

5. Let $X = \{a, b, c, d, e, f, g\}$. How many ways can you choose a subset of size 4 or 5 that contains a but does not contain g?

   Soln. Throw out g. Put a in the set you are choosing. What's left to do? Choose 3 or 4 more elements of $\{b, c, d, e, f\}$. That's two cases, so we use addition principle.

   $$\binom{5}{3} + \binom{5}{4}.$$

Another soln. Throw out g from the problem entirely, so we are choosing sets of size 4 or 5 from a set of size 6. Two cases (sizes 4 and 5). For size 4, choose a first (1 choice), then one of the five remaining (5 choices), then one of the 4 remaining (4 choices), then one of the three remaining (3 choices); by multiplication principle, there are $1 \cdot 5 \cdot 4 \cdot 3$ ways to do this. However, the three free choices (not including a) should not be ordered, so we have overcounted by 3!. Hence for this case, we have

$$\frac{5 \cdot 4 \cdot 3}{3!}.$$

The case of size 5 is similar, and we add them, for a final answer of

$$\frac{5 \cdot 4 \cdot 3}{3!} + \frac{5 \cdot 4 \cdot 3 \cdot 2}{4!}.$$

(Yes, this is the same as in the first solution.)

Another soln. In this solution, we throw out g as before. Then there are

$$\binom{6}{4} + \binom{6}{5}$$

ways to choose a subset of size 4 or 5 from the remaining letters. However, we didn't want to include the solutions that fail to include a. So we must count those and remove them. How many sets of size 4 fail to include a? That's choosing 4 from 5 things, i.e. $\binom{5}{4}$. And there is only one set of size 5 that fails to include a: $\{b, c, d, e, f\}$. Hence the answer is

$$\binom{6}{4} + \binom{6}{5} - \binom{5}{4} - 1.$$

(Yes, this is the same as in the above solutions. In fact, it is 15.)

6. You have 19 **different** gifts. How can you distribute them among three friends (Ali, Bob and Cal), so that everyone gets at least one object?

Soln. If we just want to distribute the gifts, then we can go gift-by-gift, deciding whom to give it to. My multiplication principle, that's 3^{19} possibilities.

However, we must throw out the cases where someone got nothing at all. How many ways can Ali get nothing? There are 2^{19} ways (distribute all gifts to Bob and Cal). Similarly, Bob can get nothing in 2^{19} ways, and so can Cal. So you might think we have $3^{19} - 3 \cdot 2^{19}$.

However, we have now thrown out too many cases, since we threw out the case where Cal got everything *twice* (once when counting cases where Ali got nothing, and once when counting cases where Bob got nothing). There are a few ways to fix this.

The simplest fix is to *add back in* the cases that got thrown out twice, i.e.

$$3^{19} - 3 \cdot 2^{19} + 3.$$

The idea of adding stuff back in is called *the method of inclusion-exclusion.*

Alternative fix: A different way to fix the problem is to break into cases that are truly disjoint:

(# ways to distribute gifts)
− (# ways Ali gets nothing but Bob and Cal both get something)
− (# ways Bob gets nothing but Ali and Cal both get something)
− (# ways Cal gets nothing but Ali and Bob both get something)
− (# ways Ali and Bob get nothing)
− (# ways Bob and Cal get nothing)
− (# ways Cal and Ali get nothing)
$= 3^{19} - 3 \cdot (2^{19} - 2) - 3.$

Notice that the $2^{19} - 2$ for the ways Ali gets nothing but Bob and Cal both get something is

(# ways Ali gets nothing) − (# ways Bob gets everything) − (# ways Cal gets everything) $= 2^{19} - 1 - 1.$

Both of these answers simplify to the same thing.

Postscript. We might try first giving one thing to each person, which is possible in $19 \cdot 18 \cdot 17$ ways. Then we can hand out the rest in 3^{16} ways. However, we have overcounted in this scenario. If Ali got two objects, for example and apple and a hamster, then it's possible to obtain this arrangement by choosing the apple for him first, or by choosing the hamster first. So this arrangement will be counted (at least) twice. However, the overcounting is not by a *uniform* factor in this situation, so we can't use the division principle to fix it. Starting off this way, it's not clear how to complete the problem.

1 Grading

Goals:

1. understanding of the correct context for the various counting principles (multiplication etc) and quantities (factorials, binomial coefficients etc)
2. can correctly recognize independence and overcounting
3. can synthesize the parts to a correct numerical answer
4. coherent justification and logical consistency
5. mastery of language and definitions (e.g., subsets are not ordered)
6. attention to detail and big picture (e.g., not overlooking part of the task)

Rubric:

1. A: above skills are demonstrated consistently in all standard situations, and can make reasonable progress on challenge problems
2. B: most skills demonstrated consistently, but a few skills are demonstrated only inconsistently, nevertheless with a reliable general understanding
3. C: most skills are inconsistent, but the general context is often correct, despite being misapplied, missing attention to detail, forgetting steps etc.
4. D: some demonstration of the skills in the most straightforward situations, and some evidence of big-picture understanding, if not detail
5. F: mostly incorrect applications of skills and lack of knowledge

Grades of AB or BC are in between.
On canvas, these become grades out of 10:

1. A+ = 9.5
2. A = 9
3. AB = 8.5
4. B = 8
5. BC = 7.5
6. C = 7
7. CD = 6.5
8. D = 6