# Combinatorial Proof

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A combinatorial proof is a proof that shows some equation is true by explaining why both sides count the same thing. Its structure should generally be:

Explain what we are counting.
Explain why the LHS counts that correctly.
Explain why the RHS counts that correctly.
Conclude that both sides are equal since they count the same thing.

1. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n}{n-k}.$$

   (a) What are we counting?

   (b) How does the left side count this?

   (c) How does the right side count this?

2. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

   (a) What are we counting?

   (b) How does the left side count this?

   (c) How does the right side count this?

3. Give a combinatorial proof that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

(a) What are we counting?

(b) How does the left side count this?

(c) How does the right side count this?

4.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(a) What are we counting?

(b) How does the left side count this?

(c) How does the right side count this?

5.

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

(a) What are we counting?

(b) How does the left side count this?

(c) How does the right side count this?

6. More combinatorial proofs:

(a) Actually, if you take $\binom{n}{k}$ *by definition*, as counting the number of subsets of size $k$ of a set of size $n$, then showing

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is by combinatorial proof.

(b)

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}.$$

(c)

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n}\binom{n}{0}=\binom{2n}{n}$$

(d) Invent your own!