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Let this largest integer be n.

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Suppose, for a contradiction, that there are such integers a and b. Then, $4a$ and $18b$ are even, so $4a + 18b$ is even. However, $4a + 18b = 1$, which is odd.

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Suppose, for a contradiction, that there are such integers a and b. Then, $4a$ and $18b$ are even, so $4a + 18b$ is even. However, $4a + 18b = 1$, which is odd. Since an integer cannot be both even and odd, this is a contradiction. □

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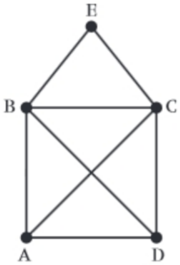
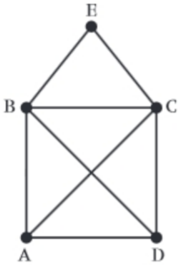
Let x be rational and non-zero and y irrational. Suppose, for a contradiction, that xy is rational. Then, since x is non-zero, $y = xy/x$ is a quotient of rational numbers. Therefore y is rational, a contradiction to our assumptions. □



Theorem

Consider the picture at left. It is impossible to traverse this diagram along the edges in a cycle (ending where you begin) using each edge exactly once.



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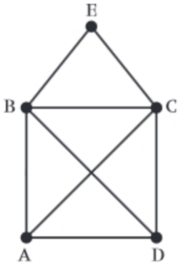

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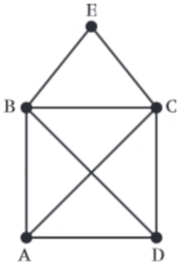
For a contradiction, suppose there were such a cycle. In such a cycle, the cycle would leave each vertex as many times as it entered it.



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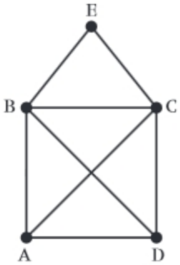

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Suppose, for a contradiction, that there were only finitely many such primes. Call the largest such prime p_0. Let p be a prime larger than p_0. Then $p + 2$ is prime. Since $p + 2 > p_0$, then $p + 4$ is prime.

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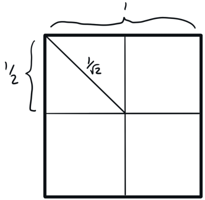
Theorem

Consider a unit square with sides of length 1*. Suppose* 5 *points are placed inside this square. Then there exist two points* x *and* y *among these five, such that the distance between* x *and* y *is less than or equal to* $1/\sqrt{2}$*.*



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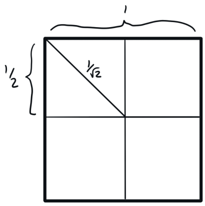
Divide the square into four quadrants as shown.



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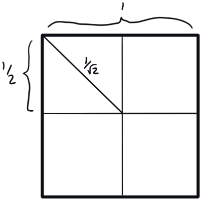

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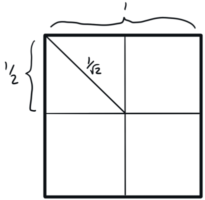

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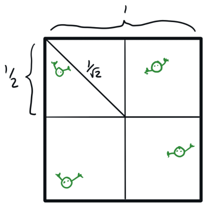

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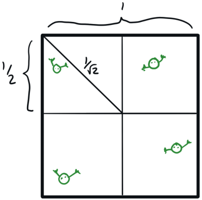

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