Math 2001, Spring 2023. Katherine E. Stange.

1 Assignment

Prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Hint: This is a "combinatorial proof" like the one in last class. The left side counts subsets of size k of a set of size n. For the right side, break this same counting problem into two cases by fixing one element of the larger set and counting subsets that contain it and those that do not. Reach out if you want more of a hint!

2 A combinatorial proof

We will show that both sides of the equation count the number of subsets of size k of a set of size n. On the left, this is by definition.

On the right, we divide the count into two cases. Let us call the set X and fix one element of X, say x. We will condition our count of subsets on whether x is in the subset.

If x is to be in the subset, then we must choose $k-1$ other elements from amongst the remaining $n-1$ elements of X, which can be done in $\binom{n-1}{k-1}$ ways.

If x is to be left out of the subset, then we must choose k other elements from amongst the remaining $n-1$ elements of X, which can be done in $\binom{n-1}{k}$ ways.

Thus, the total count, by the addition principle for cases, is

$$\binom{n-1}{k-1} + \binom{n-1}{k}.$$

3 A combinatorial proof – another variation on wordings

We will show that both sides of the equation count the same thing, hence are equal.

The count in question is the number of subsets of size k of a set of size n (call the larger set X).

The left side counts this by the definition of the binomial coefficient.

To show the right side counts this, we divide the count into two cases. Fix an element $x \in X$.

In case one, we include x in the subset. If x is in the subset, then we must choose $k-1$ other elements from amongst the remaining $n-1$ elements of X. This can be done in $\binom{n-1}{k-1}$ ways.

In case two, we do not include x in the subset. If x is not in the subset, then we must choose all k of our elements from amongst the other $n-1$ elements of X. This can be done in $\binom{n-1}{k}$ ways.

By the addition principle, the number of subsets of X of size k is the quantity given on the right hand side.

4 A purely algebraic proof

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{(k)!(n-1-k)!} \\
&= \frac{(n-1)!(k)}{(k)!(n-k)!} + \frac{(n-1)!(n-k)}{(k)!(n-k)!} \\
&= \frac{(n-1)!(k+n-k)}{(k)!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$