Induction Proofs, IV: Fallacies and pitfalls

By now, induction proofs should feel routine to you, to the point that you could almost do them in your sleep. However, it is important not to become complacent and careless, for example, by skipping seemingly minor details in the write-up, omitting quantifiers, or neglecting to check conditions and hypotheses.

Below are some examples of false induction proofs that illustrate what can happen when some minor details are left out. In each case, the statement claimed is clearly nonsensical (e.g., that all numbers are equal), but the induction argument sounds perfectly fine, and in some cases the error is quite subtle and hard to spot. Try to find it!

Example 1

Claim: $\boxed{\text{For all } n \in \mathbb{N},\ (*)\ \sum_{i=1}^{n} i = \frac{1}{2}(n + \frac{1}{2})^2}$

Proof: We prove the claim by induction.

Base step: When $n = 1$, $(*)$ holds.

Induction step: Let $k \in \mathbb{N}$ and suppose $(*)$ holds for $n = k$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{1}{2}\left(k + \frac{1}{2}\right)^2 + (k+1) \quad \text{(by ind. hypothesis)} \\
&= \frac{1}{2}\left(k^2 + k + \frac{1}{4} + 2k + 2\right) \quad \text{(by algebra)} \\
&= \frac{1}{2}\left(\left(k + 1 + \frac{1}{2}\right)^2 - 3k - \frac{9}{4} + k + \frac{1}{4} + 2k + 2\right) \quad \text{(more algebra)} \\
&= \frac{1}{2}\left((k+1) + \frac{1}{2}\right)^2 \quad \text{(simplifying).}
\end{aligned}$$

Thus, $(*)$ holds for $n = k + 1$, so the induction step is complete.

Conclusion: By the principle of induction, $(*)$ holds for all $n \in \mathbb{N}$.

Example 2

Claim: $\boxed{\text{All real numbers are equal.}}$

Proof: To prove the claim, we will prove by induction that, for all $n \in \mathbb{N}$, the following statement holds:

$(P(n))$ For any real numbers $a_1, a_2, \dots, a_n$, we have $a_1 = a_2 = \cdots = a_n$.

Base step: When $n = 1$, the statement is trivially true, so $P(1)$ holds.

Induction step: Let $k \in \mathbb{N}$ be given and suppose $P(k)$ is true, i.e., that any k real numbers must be equal. We seek to show that $P(k+1)$ is true as well, i.e., that any $k+1$ real numbers must also be equal.

Let $a_1, a_2, \dots, a_{k+1}$ be given real numbers. Applying the induction hypothesis to the first k of these numbers, $a_1, a_2, \dots, a_k$, we obtain

$$a_1 = a_2 = \cdots = a_k. \tag{1}$$

Similarly, applying the induction hypothesis to the last k of these numbers, $a_2, a_3, \dots, a_k, a_{k+1}$, we get

$$a_2 = a_3 = \cdots = a_k = a_{k+1}. \tag{2}$$

Combining (1) and (2) gives

$$a_1 = a_2 = \cdots = a_k = a_{k+1}, \tag{3}$$

so the numbers $a_1, a_2, \dots, a_{k+1}$ are equal. Thus, we have proved $P(k+1)$, and the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. Thus, any n real numbers must be equal.

Example 3

Claim: $\boxed{\text{For every nonnegative integer } n, \ (*)\ 5n = 0.}$

Proof: We prove that $(*)$ holds for all $n = 0, 1, 2, \ldots$, using strong induction with the case $n = 0$ as base case.

Base step: When $n = 0$, $5n = 5 \cdot 0 = 0$, so $(*)$ holds in this case.

Induction step: Suppose $(*)$ is true for all integers n in the range $0 \le n \le k$, i.e., that for all integers in this range $5n = 0$. We will show that $(*)$ then holds for $n = k + 1$ as well, i.e., that $(**)$ $5(k + 1) = 0$.

Write $k + 1 = i + j$ with integers i, j satisfying $0 \le i, j \le k$. Applying the induction hypothesis to i and j, we get $5i = 0$ and $5j = 0$. Then

$$5(k + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0,$$

proving $(**)$. Hence the induction step is complete.

Conclusion: By the principle of strong induction, $(*)$ holds for all nonnegative integers n.

Example 4

Claim: $\boxed{\text{For every nonnegative integer } n, \ (*)\ 2^n = 1.}$

Proof: We prove that $(*)$ holds for all $n = 0, 1, 2, \ldots$, using strong induction with the case $n = 0$ as base case.

Base step: When $n = 0$, $2^0 = 1$, so $(*)$ holds in this case.

Induction step: Suppose $(*)$ is true for all integers n in the range $0 \le n \le k$, i.e., assume that for all integers in this range $2^n = 1$. We will show that $(*)$ then holds for $n = k + 1$ as well, i.e., that $(**)$ $2^{k+1} = 1$.

We have

$$\begin{aligned}
2^{k+1} &= \frac{2^{2k}}{2^{k-1}} \quad \text{(by algebra)} \\
&= \frac{2^k \cdot 2^k}{2^{k-1}} \quad \text{(by algebra)} \\
&= \frac{1 \cdot 1}{1} \quad \text{(by strong ind. hypothesis applied to each term)} \\
&= 1 \quad \text{(simplifying)},
\end{aligned}$$

proving $(**)$. Hence the induction step is complete.

Conclusion: By the principle of strong induction, $(*)$ holds for all nonnegative integers n.

Example 5

Claim: $\boxed{\text{All positive integers are equal}}$

Proof: To prove the claim, we will prove by induction that, for all $n \in \mathbb{N}$, the following statement holds:

$(P(n))$ For any $x, y \in \mathbb{N}$, if $\max(x, y) = n$, then $x = y$.

(Here $\max(x, y)$ denotes the larger of the two numbers x and y, or the common value if both are equal.)

Base step: When $n = 1$, the condition in $P(1)$ becomes $\max(x, y) = 1$. But this forces $x = 1$ and $y = 1$, and hence $x = y$.

Induction step: Let $k \in \mathbb{N}$ be given and suppose $P(k)$ is true. We seek to show that $P(k + 1)$ is true as well.

Let $x, y \in \mathbb{N}$ such that $\max(x, y) = k+1$. Then $\max(x-1, y-1) = \max(x, y)-1 = (k + 1) - 1 = k$. By the induction hypothesis, it follows that $x - 1 = y - 1$, and therefore $x = y$. This proves $P(k + 1)$, so the induction step is complete.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. In particular, since $\max(1, n) = n$ for any positive integer n, it follows that $1 = n$ for any positive integer n. Thus, all positive integers must be equal to 1