Quantum Gates Exercise – Math 4440

1. Here's a new gate called the square root of not (presented in the $|0\rangle$, $|1\rangle$ basis as usual):

$$\sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

   Do the following:

   (a) Check that this gate is unitary.

   (b) Check that it squares to the Pauli X or NOT gate.

   (c) Figure out what $\sqrt{X}\,|0\rangle$ is in terms of $|0\rangle$ and $|1\rangle$.

   (d) Figure out what $\sqrt{X}\,|1\rangle$ is in terms of $|0\rangle$ and $|1\rangle$.

   Solution.

   (a) The conjugate transpose is

$$\sqrt{X}^{\dagger} = \frac{1}{4}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$$

   It is useful to compute

$$(1-i)^2 = -2i, \quad (1+i)^2 = 2i, \quad (1-i)(1+i) = 2.$$

   We have

$$\sqrt{X}\sqrt{X}^{\dagger} = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

   Similarly, $\sqrt{X}^{\dagger}\sqrt{X} = I$.

   (b) We compute $\sqrt{X}^2 = X$ by matrix multiplication similarly to the last part.

   (c) Since the gate is presented in the $|0\rangle$, $|1\rangle$ basis, the first column tells us where $|0\rangle$ goes, namely,

$$\sqrt{X}\,|0\rangle = \frac{1+i}{2}\,|0\rangle + \frac{1-i}{2}\,|1\rangle\,.$$

   Similarly,

$$\sqrt{X}\,|1\rangle = \frac{1-i}{2}\,|0\rangle + \frac{1+i}{2}\,|1\rangle\,.$$

2. Verify by direct computation that $R_{x,\pi/2}\,|1\rangle = -i\,|i\rangle$.

   Solution.

   The first column dictates that

$$R_{x,\pi/2}\,|1\rangle = -i\sin(\pi/4)\,|0\rangle + \cos(\pi/4)\,|1\rangle = \frac{-i}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle = -i\left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle\right) = -i\,|i\rangle\,.$$

3. Determine a 2-qubit quantum circuit that will, on input $|00\rangle$, produce output

$$\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle.$$

Solution

Apply a Hadamard gate to each qubit individually. We obtain

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

which is the state desired.

Another way to solve this problem is to find a 4×4 unitary gate which has first column consisting all of the same entry (say, all 1's up to normalization). There are various ways one may do this, including for example:

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

Note that these different solutions may not behave the same way on other inputs.

4. Define a 3-qubit gate that can be used to compute a reversible form of OR.

Solution.

We use the same trick as for AND: we use $x, y, z \mapsto z, y, z \oplus (x \vee y)$. Then on the inputs $x, y, 0$ it gives $x, y, x \vee y$.

As a matrix on the basis

$$|000\rangle, |010\rangle, |100\rangle, |110\rangle, |001\rangle, |011\rangle, |101\rangle, |111\rangle,$$

this becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$