Fourier Exercise – Math 4440

For reference, here is the QFT written in quantum notation:

$$\sum_{x=0}^{m-1} \alpha_x |x\rangle \mapsto \frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} \sum_{k=0}^{m-1} \alpha_k \omega_m^{kx} |x\rangle .$$

1. Write down the QFT matrix of dimension 8×8. You can use the 8-th root of unity notation $\omega_8 = e^{i\pi/4}$. But simplify it so that the entries are all from the set $\{1, -1, i, -i, \omega_8, -\omega_8, i\omega_8, -i\omega_8\}$.

2. The next several questions are a progression.

   (a) What do you get if you apply the QFT of size $m = 2^n$ to the state

   $$\frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} |x\rangle?$$

   Give the answer in the form

   $$\sum_{x=0}^{m-1} \alpha_x |x\rangle .$$

   where α_x is as explicit as possible.

   (b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

   (c) What about when $m = 8$?

   (d) What do you notice? Conjecture what happens in general.

   (e) Prove it. (Note, there's a complex-numbers geometric/computational proof and a two-line linear algebra proof.)

3. The next several questions are a progression.

   (a) What do you get if you apply the QFT of size $m = 2^n$ to the state

   $$\frac{1}{\sqrt{m/2}} \sum_{x=0}^{m/2-1} |2x\rangle?$$

   Give the answer in the form

   $$\sum_{x=0}^{m-1} \alpha_x |x\rangle .$$

   where α_x is as explicit as possible.

   (b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

   (c) What about when $m = 8$?

   (d) What do you notice? Conjecture what happens in general.

   (e) Prove it.