Fourier Exercise – Math 4440

1. Write down the QFT matrix of dimension 8×8. You can use the 8-th root of unity notation $\omega_8 = e^{i\pi/4}$. But simplify it so that the entries are all from the set $\{1, -1, i, -i, \omega_8, -\omega_8, i\omega_8, -i\omega_8\}$.

   Solution.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & i & i\omega & -1 & -i\omega & -i & -\omega \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & i\omega & -i & \omega & -1 & -\omega & i & -i\omega \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -i\omega & i & -\omega & -1 & \omega & -i & i\omega \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -\omega & -i & -i\omega & -1 & i\omega & i & \omega \end{pmatrix}$$

2. The next several questions are a progression.

   (a) What do you get if you apply the QFT of size $m = 2^n$ to the state

$$\frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} |x\rangle ?$$

   Give the answer in the form

$$\sum_{x=0}^{m-1} \alpha_x |x\rangle .$$

   where α_x is as explicit as possible.

   (b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

   (c) What about when $m = 8$?

   (d) What do you notice? Conjecture what happens in general.

   (e) Prove it. (Note, there's a complex-numbers geometric/computational proof and a two-line linear algebra proof.)

   Solution.

   (a)

$$\frac{1}{m} \sum_{y=0}^{m-1} \sum_{k=0}^{m-1} \omega^{kx} |x\rangle .$$

   (b) We are essentially adding up the entries to each row of the matrix, and all but the first vanish. We get

$$|0\rangle .$$

(c) Similarly, $|0\rangle$.

(d) It will always be $|0\rangle$.

(e) The two-line proof is that the QFT matrix is invertible. We are asking for $\mathbf{x}$ in the equation $F\mathbf{v} = \mathbf{x}$, but the inverse of the QFT is its conjugate transpose, $F^\dagger$, so this is $F^\dagger\mathbf{x} = \mathbf{v}$, i.e. write $\mathbf{v}$ as a sum of columns of $F^\dagger$; but the first column *is* $\mathbf{v}$, so the answer is $(1, 0, 0, 0, \ldots, 0)$, i.e. $|0\rangle$.

A more computational proof is to find the sum

$$\sum_{y=0}^{m-1}\sum_{k=0}^{m-1}\omega^{kx}.$$

Since ω is a root of unity, if x is non-zero, then this is a sum of the m m-th roots of unity, which surround the origin symmetrically and average to 0. (You can use trig to do this explicitly.) But if $x = 0$, then this is a sum of 1's.

3. The next several questions are a progression.

(a) What do you get if you apply the QFT of size $m = 2^n$ to the state

$$\frac{1}{\sqrt{m/2}}\sum_{x=0}^{m/2-1}|2x\rangle?$$

Give the answer in the form

$$\sum_{x=0}^{m-1}\alpha_x|x\rangle.$$

where α_x is as explicit as possible.

(b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

(c) What about when $m = 8$?

(d) What do you notice? Conjecture what happens in general.

(e) Prove it.

Solution.

(a)

$$\frac{\sqrt{2}}{m}\sum_{y=0}^{m-1}\sum_{k=0}^{m/2-1}\omega^{2kx}|x\rangle.$$

(b) We are essentially adding up every second entry in each row of the matrix, and all but the first and second vanish. We get

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|2\rangle.$$

(c)

$$\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|4\right\rangle .$$

(d) It will always be $\frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|4\right\rangle$.

(e) We are asking for $\mathbf{x}$ in the equation $F\mathbf{v} = \mathbf{x}$, but the inverse of the QFT is its conjugate transpose, $F^\dagger$, so this is $F^\dagger\mathbf{x} = \mathbf{v}$, i.e. write $\mathbf{v}$ as a sum of columns of $F^\dagger$; but the first column plus second column add to $\mathbf{v}$, so the answer is as conjectured.

A more computational proof is to find the sum

$$\sum_{y=0}^{m-1}\sum_{k=0}^{m/2-1} \omega^{2kx}.$$

Since ω is a root of unity, if x is non-zero mod m, then this is a sum of the m m-th roots of unity, which surround the origin symmetrically and average to 0. (You can use trig to do this explicitly.) But if $x = 0$ or $m/2$, then this is a sum of 1's.