Quantum Entanglement Exercise – Math 4440

Here is a 2-qubit state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle .$$

1. Suppose you measure the first qubit and obtain a measurement of $|0\rangle$.

   (a) What is the state after this measurement?

   Solution

   The classical state $|11\rangle$ drops off, and we obtain, after normalizing by the length

   $$\frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|01\rangle\right) = \frac{\sqrt{2}}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle .$$

   (b) If you now measure the second qubit, what are the possible results and probabilities of those results?

   Solution

   We have probability 2/3 that we obtain 00 and probability 1/3 that we obtain 01.

   Fortunately, these add up to 1 (always a good check to do).

2. Suppose instead you measure the first qubit and obtain a measurement of $|1\rangle$.

   (a) What is the state after this measurement?

   Solution

   The classical state $|11\rangle$ is the only one surviving, and we obtain, after normalizing by the length

   $$|11\rangle .$$

   (b) If you now measure the second qubit, what are the possible results and probabilities of those results?

   Solution

   We have probability 1 that we obtain 11.

3. Is this an entangled state? Justify.

   Solution

   Yes.

   One way to see this is that in the two different outcomes to measuring the first qubit, we get differing probabilities for the outcome of the second. Hence they are not independent.

Another way to see this is to try to solve the system of equations that would give rise to the 2-qubit amplitudes as a result of two single-qubit states:

$$\frac{1}{\sqrt{2}} = a_1a_2, \quad \frac{1}{2} = a_1b_2, \quad 0 = b_1a_2 \quad \frac{1}{2} = b_1b_2.$$

But this is harder in general (here it's not bad at all).

Suppose you have $n+1$ qubits. We will write $|\vec{x}\rangle$ to mean the n-qubit classical state given by the number x in binary. So for example, if $n = 2$ then $\left|\vec{0}\right\rangle = |00\rangle$, $\left|\vec{3}\right\rangle = |11\rangle$ etc. Suppose the qubits are in this state:

$$|\Phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |\vec{x}\rangle \, |x \mod 2\rangle$$

1. What is the resulting state if we measure the last qubit and obtain $|0\rangle$? Make sure you have normalized your state.

   Solution

   All the states with last qubit $|1\rangle$ disappear from the summation. That leaves exactly half of them, i.e. those where x is even. Therefore we have

   $$\frac{1}{\sqrt{2^{n-1}}} \sum_{x=0}^{2^{n-1}-1} \left|\overrightarrow{2x}\right\rangle |0\rangle \, .$$

   Notice that the length of the vector has changed (it has half as many non-zero entries), so the scalar out front has altered. That's the normalization.

2. What is the resulting state if we measure the last qubit and obtain $|1\rangle$? Make sure you have normalized your state.

   Solution All the states with last qubit $|0\rangle$ disappear from the summation. That leaves exactly half of them, i.e. those where x is odd. Therefore we have

   $$\frac{1}{\sqrt{2^{n-1}}} \sum_{x=0}^{2^{n-1}-1} \left|\overrightarrow{2x+1}\right\rangle |1\rangle \, .$$