The pigeonhole principle

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1 Pigeonhole principle

In its simplest form, here is the pigeonhole principle:

Theorem 1. *Suppose there are $n+1$ pigeons to be placed into n holes. Then there will be at least one hole that has at least two pigeons.*

1. Write down a proof of this theorem.

2. Suppose there are $2n+1$ pigeons placed into n holes. What can you conclude? Write down a proof.

3. Suppose there are 5 pigeons placed into 2 holes. Consider the statement "At least one hole has at least k pigeons". For what k is this necessarily true? For what k is it false? For the smallest k for which it is false, give a counterexample.

4. Prove it for the largest k for which it is true. (Please state your theorem and provide a proof.)

2 Pigeonhole problems

1. How many socks do you need to pick out of a drawer containing n identical white socks and n identical black socks, before you are sure you have a pair? Write a proof. What are the pigeons and what are the holes?

2. How many numbers do you need to pick from $\{1, 2, \ldots, 8\}$ before you are certain to have at least two that add together to 9? Write a proof. What are the pigeons and what are the holes?

3. Consider the integers $\{1, 2, \ldots, 2n\}$. How many must you select before you are sure you have selected at least two that are consecutive? Write a proof. What are the pigeons and what are the holes?

4. Contemplate: Are there at least two people in new york city who are not bald and have the exact same number of hairs on their heads?

5. Prove: Suppose the complete graph on $n \geq 6$ vertices has its edges coloured red or blue. Prove that the graph contains a monochromatic triangle. Hint: look at one vertex.

6. Consider this trick: A magician asks an audience member to pick five cards, which are not shown to the magician. The magician's accomplice looks at the cards, picks four of the cards and shows these four to the magician in an order of his choosing. The magician then correctly guesses the fifth card. Explain why the pigeonhole principle guarantees that such a trick is possible, and then try to come up with a good way to do it. Perform it on your classmates. Can you do it with showing three cards to guess two?