Walking the Hypercube

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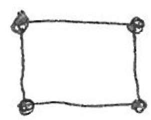
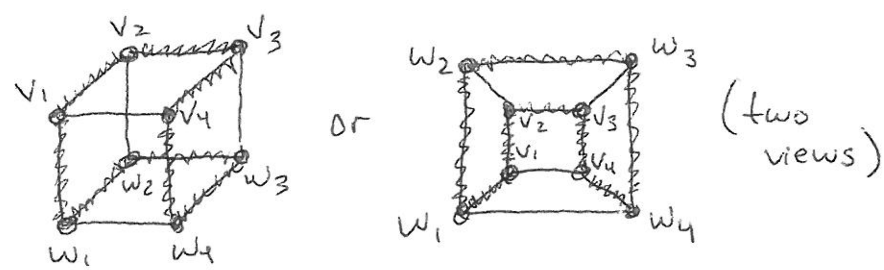
Theorem 1. *Any hypercube of dimension $n \geq 2$ has a Hamiltonian cycle.*

Proof by regular induction, standard base case. We will prove this by induction.
Base case: The 2-dimensional cube, shown here:



has four edges connected in a cycle; this is itself the Hamiltonian cycle.
Inductive step: We will assume that an n-dimensional hypercube has a Hamiltonian cycle, say $v_1, \ldots, v_{2^n}$. The $n+1$-dimensional hypercube C_{n+1} is formed from two n-dimensional hypercubes, say C_n with vertices v_i and D_n with verties w_i respectively, for $i = 1, \ldots, 2^n$. Then to the union of C_n and D_n, we add edges connecting v_i to w_i for each i, to form the $n+1$-dimensional hypercube C_{n+1}. The case of a 3D cube formed from two squares is shown here:



Now, I claim that the cycle $v_1, \ldots, v_{2^n}, w_{2^n}, w_{2^n-1}, \ldots, w_1$ is a Hamiltonian cycle. This is shown in the diagram above. We must check that all vertices appear and no vertices are repeated (this is immediate), and also that every vertex is adjacent to the previous one. For most edges, the edges are contained in either C_n or D_n and this follows from the inductive hypothesis. The only edges that are new are joining the pairs (v_{2^n}, w_{2^n}) and (w_1, v_1). But these exist from the construction of C_{n+1}. Hence this is a Hamiltonian cycle and we are done. $\square$