Worksheet on Generating Functions

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This worksheet is adapted from notes/exercises by Nat Thiem.

1 Derivatives of Generating Functions

1. If the sequence $a_0, a_1, a_2, \ldots$ has ordinary generating function $A(x)$, then what sequence has ordinary generating function $A'(x)$? By assumption,

   $$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

   Its derivative is (notice how the constant term drops off)

   $$\begin{aligned} A'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \ldots \end{aligned}$$

   Therefore, the new sequence is $b_n = (n+1)a_{n+1}$, i.e.

   $$b_0 = a_1, b_2 = 2a_2, b_3 = 3a_3, \ldots$$

2. Compute the derivative of $\frac{1}{1-x}$ with respect to x (this is a pure calculus question).

   Answer: $\frac{1}{(1-x)^2}$

3. Now expand the result as an infinite series in powers of x.

   This uses Binomial Theorem:

   $$(1-x)^{-2} = \sum_{n \geq 0} \binom{-2}{n} (-x)^n = \sum_{n \geq 0} \binom{-2}{n} (-1)^n x^n.$$

4. Combine the last three parts to prove that

   $$\binom{-2}{n} (-1)^n = (n+1).$$

   (note: this can be proven more directly; the point is to illustrate the use of generating functions)

   Let $A(x) = \frac{1}{1-x}$ which we know to be the generating function of the sequence $a_n = 1$ (the sequence of all 1's).

By the first part, $A'(x)$ is the generating function of the sequence $b_n = n$, i.e. $1, 2, 3, 4, \cdots$ (the integers). Therefore it has series expansion

$$A'(x) = \sum_{n \geq 0} (n+1)x^n.$$

(Note how the x^0 term is 1 etc.)

By the second part, we know that $A'(x)$ has series expansion

$$A'(x) = \sum_{n \geq 0} \binom{-2}{n} (-1)^n x^n.$$

Comparing coefficients of x^n, we obtain

$$\binom{-2}{n} (-1)^n = (n+1).$$

5. If the sequence $e_0, e_1, e_2, \ldots$ has exponential generating function $E(x)$, then what sequence has exponential generating function $E'(x)$? By assumption,

$$E(x) = \sum_{n \geq 0} e_n \frac{x^n}{n!}.$$

Taking derivatives,

$$\begin{aligned} E'(x) &= \sum_{n \geq 1} n e^n \frac{x^{n-1}}{n!} \\ &= \sum_{n \geq 1} \frac{n e^n}{n} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n \geq 1} e^n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{m \geq 0} e^{m+1} \frac{x^m}{m!}. \end{aligned}$$

Notice how I went to the trouble to write the result in the form

$$\sum_{n \geq 0} (\text{stuff}) \frac{x^n}{n!}$$

again. When we put it in that form, it is the exponential generating function of the 'stuff'. That is, $E'(x)$ is the exponential generating function of the sequence

$$f_n = e^{n+1}.$$

In other words, the sequence is simply shifted.

2 Products of Ordinary Generating Functions

1. Suppose $A(x)$ is the ordinary generating function for $a_0, a_1, a_2, \ldots$ and $B(x)$ is the ordinary generating function for $b_0, b_1, b_2, \ldots$. Write down the sequence having $A(x)B(x)$ as ordinary generating function.

   Let
   $$A(x) = \sum_{n\geq 0} a_n x^n, \quad B(x) = \sum_{n\geq 0} b_n x^n.$$
   Then
   $$A(x)B(x) = \left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{m\geq 0} b_m x^m\right).$$
   To deal with this, I have to find out the coefficient of each x^k in the product. The product will have terms x^k for all $k \geq 0$. To make such a term, we have to combine an $a_n x^n$ and a $b_m x^m$ such that $n + m = k$. There are various ways to do this. We have
   $$A(x)B(x) = \sum_{k\geq 0}(a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0)x^k = \sum_{k\geq 0}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)x^k.$$
   The stuff in the interior brackets is the sequence, which is
   $$c_n = \sum_{i=0}^{k} a_i b_{k-i}.$$

2. Given an ordinary generating function $A(x)$ for a sequence $a_0, a_1, a_2, \ldots$, what sequence has ordinary generating function $\frac{1}{1-x}A(x)$? We can do this by appealing to the previous problem. We have $B(x) = \frac{1}{1-x}$, i.e. $b_n = 1$. So it is
   $$\sum_{i=0}^{k} a_i.$$

3 Products of Exponential Generating Functions

1. Suppose $E(x)$ is the exponential generating function for $e_0, e_1, e_2, \ldots$ and $F(x)$ is the exponential generating function for $f_0, f_1, f_2, \ldots$. Write down the sequence having $E(x)F(x)$ as exponential generating function.

   Here, the main challenge is just remembering to put it in 'exponential form' at the end. We are taking a product of
   $$E(x)F(x) = \left(\sum_{n\geq 0} e_n \frac{x^n}{n!}\right)\left(\sum_{m\geq 0} f_m \frac{x^m}{m!}\right).$$

Therefore, as before or by calling on the before (with $a_n = e_n/n!$ and $b_n = f_n/n!$), we obtain

$$E(x)F(x) = \sum_{k\geq 0}\left(\sum_{i=0}^{k} \frac{e_i}{i!}\frac{f_{k-i}}{(k-i)!}\right)x^k.$$

Now we just have to massage this back into exponential form (because the question asked what sequence has $E(x)F(x)$ as its *exponential* generating function).

$$\begin{aligned} E(x)F(x) &= \sum_{k\geq 0}\left(\sum_{i=0}^{k} \frac{e_i}{i!}\frac{f_{k-i}}{(k-i)!}\right)x^k \\ &= \sum_{k\geq 0}\left(\sum_{i=0}^{k} \frac{e_i}{i!}\frac{f_{k-i}}{(k-i)!}k!\right)\frac{x^k}{k!} \\ &= \sum_{k\geq 0}\left(\sum_{i=0}^{k} \binom{k}{i} e_i f_{k-i}\right)\frac{x^k}{k!} \end{aligned}$$

Therefore the final sequence is

$$\sum_{i=0}^{k}\binom{k}{i}e_i f_{k-i}.$$

2. Suppose $E(x)$ is the exponential generating function for a sequence $e_0, e_1, e_2, \ldots$. What sequence has generating function $e^x E(x)$?

   We can apply the last part, with $F(x) = e^x$ which is the exponential generating function of the sequence of all 1's. Therefore $f_i = 1$ and we have
   $$\sum_{i=0}^{k}\binom{k}{i}e_i.$$

3. Use the last problem to figure out what sequence has $\frac{e^x}{1-x}$ as its **exponential** generating function. This means setting $E(x) = \frac{1}{1-x}$. Since

   $$\frac{1}{1-x} = \sum_{n\geq 0} x^n = \sum_{n\geq 0} n!\frac{x^n}{n!}$$

   we see that the sequence $e_n = n!$ has $\frac{1}{1-x}$ as its exponential generating function. So using the product formula,

   $$\sum_{i=0}^{k}\binom{k}{i}i!.$$

This can be rewritten, if you like, as a sum of falling factorials:

$$\sum_{i=0}^{k} (k)_i.$$

4. Show that $2^n = \sum_{m=0}^{n} \binom{n}{m}$. Hint: Compute e^{2x} as a series directly and as a product of known generating functions, and compare.

   Computing e^{2x} directly Since $e^x = \sum_{n \geq 0} \frac{x^n}{n!}$, we have

   $$e^{2x} = \sum_{n \geq 0} \frac{(2x)^n}{n!} = \sum_{n \geq 0} 2^n \frac{x^n}{n!}.$$

   Computing it as a product of generating functions The product is $e^x e^x$. We are using the product formula with $e_n = 1$ and $f_n = 1$, so we obtain a new series

   $$e^x e^x = \sum_{n \geq 0} \left(\sum_{m=0}^{n} \binom{n}{m} \right) \frac{x^n}{n!}.$$

   Comparing coefficients in the two expressions for the same series, we obtain

   $$2^n = \sum_{m=0}^{n} \binom{n}{m}.$$

4 An example we know

1. What sequence has ordinary generating function $\frac{1}{(1-x)^k}$? It is a sequence we have studied in this class.

   First method. This is a k-fold product of $1/(1-x)$. We have

   $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

   and so taking the product with itself k times:

   $$\frac{1}{1-x} \cdots \frac{1}{1-x} = (1 + x + x^2 + x^3 + \cdots) \cdots (1 + x + x^2 + x^3 + \cdots).$$

   where each product is k copies.

   To determine the coefficient of x^n in the result, we must choose one term from each of the series on the right. That is, we must choose an integer (the exponent) in each of the factors. For example, suppose $k = 3$. If we pick

   $$(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)$$

   this gives $x^2 \cdot 1 \cdot x^2 = x^4$. Here, $4 = 2+0+2$. Therefore we obtain one term x^n for each way of writing n as an ordered sum of non-negative integers.

   Therefore, this is the generating function for weak compositions of n!

2. Prove the last in another way. Hint: you could use binomial theorem, or you could use the techniques we used to describe the generating function for p_n.

   Second Method Now we can also use binomial theorem.

$$\frac{1}{(1-x)^k} = (1-x)^{-k} = \sum_{n\geq 0} \binom{-k}{n}(-x)^n. = \sum_{n\geq 0} \binom{-k}{n}(-1)^n x^n.$$

   We can now try to simplify the resulting coefficient

$$a_n = \binom{-k}{n}(-1)^n$$

   Using the definition of the generalized binomial coefficient, this is

$$\begin{aligned} a_n &= \frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}(-1)^n \\ &= (-1)^n \frac{k(k+1)\cdots(k+n-1)}{n!}(-1)^n \\ &= \frac{k(k+1)\cdots(k+n-1)}{n!} \\ &= \frac{(k+n-1)!}{n!}(k-1)! \\ &= \binom{k+n-1}{n} \end{aligned}$$

   Finally, I claim this is the number of ways of making a weak composition of n into k parts. For, such compositions are in bijection with strings of n $*$'s and $k-1$ $/$'s. For any composition $n = r_1 + r_2 + \cdots + r_k$, write r_1 $*$'s, then a $/$, then r_2 $*$'s, then a $/$ etc., ending with r_k $*$'s. For example,

$$* \; * \; / \; * \; /$$

   corresponds to $3 = 2 + 1 + 0$. To count these, we must arrange the n stars and $k-1$ lines. That is, from $n+k-1$ symbol positions, choose the n positions where we put stars, and let the rest be lines.